Information Processing Letters 110 (2010) 979–985

Contents lists available at ScienceDirect

Information Processing Letters

www.elsevier.com/locate/ipl

Greedy algorithms for generalized k-rankings of paths

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ARTICLE INFO

Article history:
Received 14 September 2009
Received in revised form 25 June 2010
Accepted 9 July 2010
Available online 10 August 2010
Communicated by S.E. Hambrusch

Keywords:
Rank number
Ordered coloring
Minimal ranking
Vertex coloring
Greedy algorithm
Graph algorithm

ABSTRACT

A k-ranking of a graph is a labeling of the vertices with positive integers $1, 2, \ldots, k$ so that every path connecting two vertices with the same label contains a vertex of larger label. An optimal ranking is one in which k is minimized. Let P_n be a path with n vertices. A greedy algorithm can be used to successively label each vertex with the smallest possible label that preserves the ranking property. We seek to show that when a greedy algorithm is used to label the vertices successively from left to right, we obtain an optimal ranking. A greedy algorithm of this type was given by Bodlaender et al. in 1998 [1] which generates an optimal k-ranking of P_n. In this paper we investigate two generalizations of rankings. We first consider bounded (k, s)-rankings in which the number of times a label can be used is bounded by a predetermined integer s. We then consider k_t-rankings where any path connecting two vertices with the same label contains t vertices with larger labels. We show for both generalizations that when G is a path, the analogous greedy algorithms generate optimal k-rankings. We then proceed to quantify the minimum number of labels that can be used in these rankings. We define the bounded rank number $_{r,s}(G)$ to be the smallest number of labels that can be used in a (k, s)-ranking and show for $n \geqslant 2$, $_{r,s}(P_n) = \lceil((n-(2^i-1))/s)\rceil + i$ where $i = \lfloor \log_2(s) \rfloor + 1$. We define the k_t-rank number, $^t_r(G)$ to be the smallest number of labels that can be used in a k_t-ranking. We present a recursive formula that gives the k_t-rank numbers for paths, showing $^t_r(P_j) = n$ for all $a_{n-1} < j \leqslant a_n$ where $\{a_n\}$ is defined as follows: $a_1 = 1$ and $a_n = \lfloor((t+1)/t)a_{n-1}\rfloor + 1$.

1. Introduction

A labeling $f : V(G) \to \{1, 2, \ldots, k\}$ is a *k-ranking* of a graph G if every path between two vertices with the same label contains a vertex with a larger label. Hence k-rankings are vertex colorings with an additional condition imposed. An optimal ranking is one where k is minimized. Following along the lines of the chromatic number, the *rank number of a graph* $\chi_r(G)$ is defined to be the smallest k such that G has a k-ranking. An example of a k-ranking is given in Fig. 1.

Rankings have been studied extensively (see [4] and the survey [6]). Early studies involving the rank number of a graph were motivated by its applications in the design of very large scale integration (VLSI) layout and the Cholesky factorizations associated with parallel processing (see [1,2,7]).

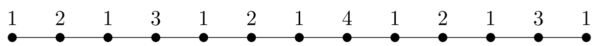

Fig. 1. A k-ranking of P_{13}.

Optimal rankings of $P_n = v_1, v_2, \ldots, v_n$ such as the one given in Fig. 1 can be constructed by labeling v_i with $\gamma + 1$ where 2^γ is the largest power of 2 that divides i [1]. It was also shown in [1] that $\chi_r(P_n) = \lfloor \log_2(n) \rfloor + 1$. We will refer to this ranking as the *standard ranking*. This optimal ranking can be obtained using a greedy algorithm where each successive vertex is given the smallest possible label that does not violate the ranking property.

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doi:10.1016/j.ipl.2010.07.011

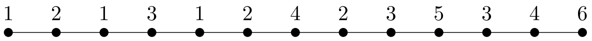

Fig. 2. A bounded (k, s)-ranking of P_{13} with $s = 3$.

We consider two generalizations of rankings, and show for both cases when G is a path, greedy implies optimal. We first consider *bounded* (k, s)-rankings, where the number of times a label can be used is bounded by a predetermined integer s. This idea was proposed by Dereniowski [3] for its relation to parallel processing where the number of machines is limited. We let $\chi_{r,s}(G)$ be the smallest integer k such that G has a k-ranking using each label at most s times. For paths, a greedy algorithm labels each successive vertex with the smallest possible label that preserves both the ranking and bounded properties. An example of a bounded (k, s)-ranking is given in Fig. 2. We show that when G is a path, the greedy algorithm generates an optimal bounded ranking. Furthermore, we show $\chi_{r,s}(P_n) = \lceil \frac{n-(2^i-1)}{s} \rceil + i$ where $i = \lfloor \log_2(s) \rfloor + 1$.

We then consider k_t*-rankings* where any path connecting two vertices of the same rank contains at least t vertices of larger ranks. An example is given in Fig. 2. The k_t*-rank number of a graph* $\chi_r^t(G)$ is defined to be the smallest k such that G has a minimal k_t-ranking. Hence rankings are known as k_1-rankings and $\chi_r^1(G) = \chi_r(G)$. We show for this generalization when G is a path the greedy algorithm generates an optimal k_t-ranking. We show $\chi_r^t(P_j) = n$ for all $a_{n-1} < j \leqslant a_n$ where $\{a_n\}$ is defined as follows: $a_1 = 1$ and $a_n = \lfloor \frac{t+1}{t} a_{n-1} \rfloor + 1$.

2. Bounded (k, s)-rankings

Recall that a bounded (k, s)-ranking is a k-ranking of a graph in which each label is used at most s times for some fixed positive integer s. We present a monotonicity property for bounded rankings.

Lemma 1. *Let $m < n$. Then $\chi_{r,s}(P_m) \leqslant \chi_{r,s}(P_n)$.*

Proof. Let f be a bounded ranking of P_n using k labels. We can use this labeling restricted to the first m labels to give a bounded ranking of P_m using less than or equal to k labels. □

We will first establish the following upper bound.

Lemma 2. *Let $n \geqslant 2$. Then $\chi_{r,s}(P_n) \leqslant \lceil \frac{n-(2^i-1)}{s} \rceil + i$ where $i = \lfloor \log_2(s) \rfloor + 1$.*

Before proceeding with the proof we give an example.

Example 3. We seek to determine $\chi_{r,10}(P_{35})$.

The largest path that has a ranking with $\chi_r(P_{10}) = 4$ labels is P_{15}. The labels from this ranking serve as the first 15 labels of our bounded ranking of P_{35}. For the next block of s labels we consider the largest path we can label using $\chi_r(P_s) - 1 = 3$ labels. This is obtained using the standard ranking for this path $1, 2, 1, 3, 1, 2, 1$. We then add 1 to each of these labels to get $2, 3, 2, 4, 2, 3, 2$. After the starting block we have a surplus of $10 - 2^3 = 2$ of the label 1. We insert vertices at the first and third positions of the second block and label these vertices with a 1. This produces the labeling $1, 2, 1, 3, 2, 4, 2, 3, 2$. Finally we insert a vertex at the beginning of this block that is labeled $\chi_r(P_{10}) + 1 = 5$. This gives us the labeling $5, 1, 2, 1, 3, 2, 4, 2, 3, 2$ for the $10 - 2^3 + 2^3 - 1 + 1 = 10$ labels in the second block. For the next block of labels of vertices $26, 27, \ldots, 35$ we simply add 1 to each of the labels in the block $16, 17, \ldots, 25$. This gives us the ranking, $[1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1]$, $[5, 1, 2, 1, 3, 2, 4, 2, 3, 2]$, $[6, 2, 3, 2, 4, 3, 5, 3, 4, 3]$. Hence $\chi_{r,s}(P_n) = 6$ for $s = 10$ and $n = 35$.

Table 1
The restrictions on the quantities of each label where $s = 3$.

Label	1	2	3	4	5	6
Bound	3	3	3	3	3	3
Max	32	16	8	4	2	1

We continue with the proof.

Proof. We prove the upper bound by constructing a k-ranking where $k = \lceil \frac{n-(2^i-1)}{s} \rceil + i$ and $i = \lfloor \log_2(s) \rfloor + 1$. This will consist of a 'starting block' followed by 'blocks' each with s vertices. We first construct the starting block. These will be the labels of the maximum-sized path that can be ranked using $\chi_r(P_s)$ labels. For the next block of s labels we consider the largest path we can label using $\chi_r(P_s) - 1$ labels. We note that in our construction we will have $s - 2^{\chi_r(P_s)-1}$ vertices we can still label with the smallest label after the starting block of vertices. We then have $2^{\chi_r(P_s)-1} - 1$ labels for this path. Finally we have one label for the largest label in this block. By our construction the number of vertices in each block will be $s - 2^{\chi_r(P_s)-1} + 2^{\chi_r(P_s)-1} - 1 + 1 = s$. We note that for each block after the second we label the vertices in block i by adding 1 to the vertices in block $i - 1$. The number of labels used in the first block is $i = \lfloor \log_2(s) \rfloor + 1$ and the number of labels used in the successive blocks is $\lceil \frac{n-(2^i-1)}{s} \rceil$. Hence $\chi_{r,s}(P_n) \leqslant \lceil \frac{n-(2^i-1)}{s} \rceil + i$ where $i = \lfloor \log_2(s) \rfloor + 1$. □

For the lower bound, we begin by finding an expression $M(\alpha)$ that gives the largest path that can be labeled with a bounded (k, s)-ranking using only α labels. In the standard ranking there will be 2^{i-1} occurrences of the i-th largest label. The idea will be to take the smaller of the two bounds given by the bounded property and the occurrences of the i-th largest label. We give an example in Table 1.

In general we have the following formula for the bounded (k, s)-rank number on $n = M(\alpha)$ vertices.

$$M(\alpha) = \sum_{j=1}^{\alpha} \min\{s, 2^{\alpha-j}\}.$$

We are now ready to establish the lower bound.

Lemma 4. *Let $n \geqslant 2$. Then $\chi_{r,s}(P_n) \geqslant \lceil \frac{n-(2^i-1)}{s} \rceil + i$ where $i = \lfloor \log_2(s) \rfloor + 1$.*