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Terminal coalgebras and free iterative theories

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Abstract

Every finitary endofunctor H of **Set** can be represented via a finitary signature Σ and a collection of equations called "basic". We describe a terminal coalgebra for H as the terminal Σ-coalgebra (of all Σ-trees) modulo the congruence of applying the basic equations potentially infinitely often. As an application we describe a free iterative theory on H (in the sense of Calvin Elgot) as the theory of all rational Σ-trees modulo the analogous congruence. This yields a number of new examples of iterative theories, e.g., the theory of all strongly extensional, rational, finitely branching trees, free on the finite power-set functor, or the theory of all binary, rational unordered trees, free on one commutative binary operation.
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1. Introduction

It is well-known that for any finitary signature Σ an initial Σ-algebra I_Σ is the algebra of all finite Σ-trees, and a terminal Σ-coalgebra T_Σ is the algebra of all (finite and infinite) Σ-trees. We now

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prove the analogous statement for every finitary endofunctor H of **Set**. First, we express H as a quotient of the polynomial functor H_Σ, given by

$$H_\Sigma X = \Sigma_0 + \Sigma_1 \times X + \Sigma_2 \times X^2 + \cdots$$

for some finitary signature Σ. In fact, being finitary (i.e., preserving directed colimits) is, for set functors, equivalent to being a quotient of some H_Σ. Moreover, the quotient is expressed by a collection of *basic equations*, i.e., equations of the form

$$\sigma(x_1, \ldots, x_n) = \varrho(y_1, \ldots, y_k),$$

where σ and ϱ are operation symbols and x_i and y_i are variables (not necessarily distinct).

Example: the finite-power-set functor $\mathscr{P}_f$ is a quotient of the polynomial functor

$$H_\Sigma X = 1 + X + X^2 + \cdots$$

(of the signature Σ which has one n-ary operation σ_n for every $n \in \mathbb{N}$) via the basic equations

$$\sigma_n(x_1, \ldots, x_n) = \sigma_k(y_1, \ldots, y_k),$$

where n and k are arbitrary numbers and the variables are such that the set $\{x_1, \ldots, x_n\}$ is equal to $\{y_1, \ldots, y_k\}$.

Now given such a presentation of H, it is well known that an initial H-algebra I has the form

$$I = I_\Sigma / {\sim},$$

where $\sim$ is the congruence generated by the basic equations. That is, two finite Σ-trees t and s are congruent iff t can be obtained from s by a finite application of the basic equations. We prove below that a terminal H-coalgebra has the form

$$T = T_\Sigma / {\sim^*},$$

where $\sim^*$is the congruence of finite and infinite applications of the basic equations. The infinite application has a simple definition, inspired by the description of the terminal $\mathscr{P}_f$-coalgebra provided by Barr [14]: Given infinite Σ-trees t and s denote by $\partial_k t$ and $\partial_k s$ the trees we obtain from them by cutting them at level k. Then we define $\sim^*$ as follows:

$$t \sim^* s \qquad \text{iff} \qquad \partial_k t \sim \partial_k s \qquad \text{for all } k = 0, 1, 2, \ldots$$

Example: a terminal $\mathscr{P}_f$-coalgebra is the coalgebra of all finitely branching strongly extensional trees, i.e., finitely branching unordered trees such that distinct children of every node define non-bisimilar subtrees, see [25]. The reason is that they form a choice class of the above congruence $\sim^*$: every unordered tree is congruent to a unique strongly extensional tree.

The main result of our paper is the above description of a terminal coalgebra for any finitary set functor H. From this we (easily) derive a concrete description of a free iterative theory $\mathscr{R}_H$ on H. Iterative theories were introduced by Elgot [17] as a means of an algebraic description of infinite

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