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Analogues of Chaitin's Omega in the computably enumerable sets ☆

G. Barmpalias [a,*], R. Hölzl [b], A.E.M. Lewis [c], W. Merkle [d]

[a] State Key Lab of Computer Science, Institute of Software, Chinese Academy of Sciences, Beijing 100190, P.O. Box 8718, China
[b] Bureau 6A35 LIAFA, 175 rue du Chevaleret, 75013 Paris, France
[c] School of Mathematics, University of Leeds, LS2 9JT Leeds, United Kingdom
[d] Institut für Informatik, Ruprecht-Karls-Universität, Heidelberg, Germany

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ABSTRACT

We show that there are computably enumerable (c.e.) sets with maximum initial segment Kolmogorov complexity amongst all c.e. sets (with respect to both the plain and the prefix-free version of Kolmogorov complexity). These c.e. sets belong to the weak truth table degree of the halting problem, but not every weak truth table complete c.e. set has maximum initial segment Kolmogorov complexity. Moreover, every c.e. set with maximum initial segment prefix-free complexity is the disjoint union of two c.e. sets with the same property; and is also the disjoint union of two c.e. sets of lesser initial segment complexity.

1. Introduction

Kolmogorov complexity measures the complexity of a finite sequence in terms of the shortest program that can generate it. It may also be used in order to study the initial segment complexity of infinite sequences, and it is this approach that led to the definition of random sequences in [20,7]. Measures of relative initial segment complexity were initially introduced for the class of computably enumerable (c.e.) reals (i.e. reals that are the limits of increasing computable sequences of rationals) and were used in order to characterize Chaitin's Ω numbers as the c.e. reals with maximum initial segment complexity. In this note we are concerned with the initial segment complexity of c.e. sets. We discover a class of c.e. sets of maximum initial segment complexity and study some of its properties. These c.e. sets may be seen as analogues of Chaitin's Ω numbers in the class of c.e. sets.

In Section 1.1 we review the measures that have been used in the literature in order to classify classes of reals according to their initial segment complexity. In Section 1.2 we give an account of the known properties concerning

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* Corresponding author.
E-mail addresses: barmpalias@gmail.com (G. Barmpalias), r@hoelzl.fr (R. Hölzl), andy@aemlewis.co.uk (A.E.M. Lewis), merkle@math.uni-heidelberg.de (W. Merkle).
URLs: http://www.barmpalias.net (G. Barmpalias), http://hoelzl.fr (R. Hölzl), http://aemlewis.co.uk (A.E.M. Lewis), http://www.math.uni-heidelberg.de/logic/merkle/merkle.html (W. Merkle).

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the initial segment complexity of c.e. sets. In Section 1.3 we give an outline of our results, which are presented in detail in the main part of this note.

1.1. Measures of relative initial segment complexity

One of the earliest measures for comparing the initial segment complexity of *reals* (which we identify with their binary expansion) was introduced and studied in [26]. It is known as the 'Solovay reducibility', often denoted by $\leqslant_S$, and for c.e. reals it essentially measures the hardness of approximation 'from below'. It is a preorder and it induces a partially ordered degree structure that is known as the Solovay degrees. In a series of papers [26,8,15] it was shown that the random c.e. reals are exactly the reals in the greatest Solovay degree, and they coincide with the halting probabilities of universal prefix-free machines. This structure was studied further (see [9, Section 9.5] for an overview) and was generally accepted as an adequate measure for classifying initial segment complexities for the class of c.e. reals. A number of related measures were introduced in [10] with the hope of providing measures of relative complexity for different classes of reals. Let K_M denote the Kolmogorov complexity function with respect to the Turing machine M (i.e., $K_M(\sigma)$ is the length of the shortest string τ such that $M(\tau)=\sigma$, and ∞ if this does not exist). Let $K=K_U$ where U is a fixed optimal universal prefix-free machine and let $C=K_V$ where V is a fixed optimal universal (plain) Turing machine. Also, let $C(\sigma|\tau)$ denote the Kolmogorov complexity of σ relative to τ (i.e. when τ is given as an oracle in the underlying machine that describes σ). A real X is called random if $\exists c\,\forall n,\ \ K(X\restriction_n)\geqslant n-c$. Perhaps the most straightforward measure of relative initial segment complexity is $\leqslant_K$ (already implicit in [26]).

$$X\leqslant_K Y \quad \overset{\text{def}}{\Longleftrightarrow} \quad \exists c\,\forall n\,\bigl(K(X\restriction_n)\leqslant K(Y\restriction_n)+c\bigr). \tag{1.1}$$

We may express the fact that $X\leqslant_K Y$ simply by saying that the prefix-free initial segment complexity of X is less than (or equal to) the prefix-free initial segment complexity of Y. The plain complexity version $\leqslant_C$ of the above relation is defined analogously. These preorders induce the K-degrees and the C-degrees respectively, which have received a certain amount of attention (see [9, Section 9.7]). We note that $\leqslant_S$ is contained in $\leqslant_K$ and so the K-degrees of c.e. reals have a largest element that contains the random c.e. reals. The main proposal for an alternative to Solovay reducibility that applies to more general classes of reals was the *relative K-reducibility* (in symbols, rK), which is defined by

$$X\leqslant_{rK} Y \quad \overset{\text{def}}{\Longleftrightarrow} \quad \exists c\,\forall n\,\bigl(K(X\restriction_n\mid Y\restriction_n)\leqslant c\bigr). \tag{1.2}$$

Note that $X\leqslant_{rK} Y$ can be defined equivalently using plain complexity, by the relation $\exists c\,\forall n(C(X\restriction_n\mid Y\restriction_n)\leqslant c)$. This follows from the basic relations between plain and prefix-free complexity, namely the fact that there exists a constant d such that $C(\sigma|\tau)\leqslant K(\sigma|\tau)+d$ and $K(\sigma|\tau)\leqslant 2C(\sigma|\tau)+d$ for all strings σ,τ. It is not hard to see that the relation $X\leqslant_{rK} Y$ is equivalent to (1.3).

$$\begin{aligned}&\text{There exists a partial computable function}\\ &f:2^{<\omega}\times\omega\to 2^{<\omega}\text{ and a constant } c \text{ such that}\\ &\forall n\,\exists j<c\,\bigl(f(Y\restriction_n,j)\downarrow = X\restriction_n\bigr).\end{aligned} \tag{1.3}$$

Here ω denotes the set of natural numbers and $2^{<\omega}$ denotes the set of finite binary strings. This shows that $X\leqslant_{rK} Y$ implies $X\leqslant_K Y$ and $X\leqslant_C Y$. Moreover (as observed in [10]) $X\leqslant_{rK} Y$ implies $X\leqslant_T Y$ (where $\leqslant_T$ is the Turing reducibility). In [22] it was observed that $X\leqslant_C Y$ implies $X\leqslant_T Y$, provided that Y is a subset of $\{2^{2^n}\mid n\in\omega\}$.

1.2. The initial segment complexity of c.e. sets

By [2], if A is a c.e. set then $\exists c\,\forall n,\ C(A\restriction_n)\leqslant 2\log n+c$ (here and throughout this paper, $\log n$ denotes the largest integer which is less than or equal to the logarithm of n on base 2); on the other hand, there are c.e. sets B such that $\forall n,\ C(B\restriction_n)\geqslant \log n-b$ for some constant b. Each of these observations lead to a more informative view about c.e. sets with complicated initial segments. The first is from [16] and is known as the *Kummer dichotomy*. It says that every member in a certain class of c.e. Turing degrees that is known as the *array non-computable degrees* contains a c.e. set A such that $C(A\restriction_n)\geqslant 2\log n-c$ for infinitely many n and some constant c; on the other hand if the degree of a c.e. set B is not in that class and f is any computable order (i.e. non-decreasing unbounded function) then $\exists b\,\forall n,\ C(B\restriction_n)\leqslant \log n+f(n)+b$. The second is from [13] and [14] and characterizes the c.e. sets A such that $\forall n,\ C(A\restriction_n)\geqslant f(n)$ for a computable order f, as the weak truth table complete c.e. sets (i.e. the sets that compute the halting problem with a computable bound on their use in the computation). These are also called *complex sets*.

Further research on this topic concerns the behavior of the measures of complexity that we discussed in Section 1.1 on the class of c.e. sets. In [3] it was shown that in the Solovay degrees of c.e. sets there are pairs with no upper bound; in particular, there is no maximum degree. In [5] it was shown that there are no minimal pairs in the structure of the K-degrees of c.e. sets. This gave an elementary property that distinguishes this structure from the C-degrees, the rK-degrees and the Solovay degrees of c.e. sets. A number of other features in the K-degrees and the C-degrees of c.e. sets (including splitting theorems and cone avoidance arguments) were shown in [27, Chapter 2] (see [4, Section 5]) and [6, Section 6] by emulating the corresponding arguments in the c.e. Turing degrees.

1.3. Our results

Perhaps the most basic question concerning the relative initial segment complexity of c.e. sets is whether there exist c.e. sets that are more complex (modulo a constant) than any other c.e. set. In view of the Kummer dichotomy and the behavior of the c.e. sets in the Solovay degrees, one would guess a negative answer for any of the measures of relative complexity of Section 1.1. In Section 2 we show that, surprisingly, there are complete (i.e. maximum) elements in the partial orders of the rK-degrees, the C-degrees and the K-degrees of c.e. sets.