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Line graph operation and small worlds

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ABSTRACT

Complex networks, such as small world networks, are the focus of recent interest because of their potential as models for the interaction networks of complex systems. Most of the well-known models of small world networks are stochastic. The randomness makes it more difficult to gain a visual understanding of how networks are shaped, and how different vertices relate to each other. In this paper, we present and study a method for constructing deterministic small worlds using the line graph operator. This operator introduces cliques at every vertex of the original graph, which may imply larger clustering coefficients. On the other hand, this operator can increase the diameter at most by one and assure the small world property.

1. Introduction

The neural networks, transportation systems, biological and chemical systems, social networks, the Internet and the World Wide Web, are only a few examples of systems composed of a large number of highly interconnected dynamic units. A widely used approach for capturing global properties of large networks is to model them as graphs, whose vertices represent the objects or individuals and whose edges describe pairwise connections. Of course, this is a restrictive representation, since the interaction between two objects or individuals depends also on time, space and many other factors. From a practical point of view, such a representation provides a simple but still very informative model of the real network.

In this representation, real networks are characterized by correlations in the vertex degrees, by having relatively short paths between any two vertices, and by the presence of a large number of short cycles or specific motifs. This feature of having a relatively short path between any two vertices within a network, despite of its large size, is known as the *small world property*. It was first investigated, in the social context, by Milgram [12] in the 1960s in a series of experiments to estimate the actual number of steps in a chain of acquaintances.

The small world property has been observed in a variety of other real networks, including biological and technological ones, and is an obvious mathematical property in some network models, e.g., in random graphs. In contrast to random graphs, the small world property in real networks is often associated with the presence of clustering, indicated by high values of the clustering coefficient. For this reason, Watts and Strogatz [18] proposed to define *small world networks* as networks having both a short diameter, like random graphs, and a high clustering coefficient, like regular lattices. Thus, their model of a large network is situated between an ordered finite lattice and a random graph, presenting the small world property and high clustering coefficient. Soon after the appearance of [18], Barthélémy and Amaral [4] studied the origins of the small world behaviour, while Barrat and Weigt [3] addressed analytically as well as numerically the structure properties of the Watts–Strogatz model. Since then the study of complex

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networks, including small world networks, has experienced considerable progress as an interdisciplinary subject. Several excellent general reviews and books are available and we refer to them for the reader who would like to obtain more information on the topic [1,2,5,13,15–17]. In 2000, Kleinberg [10] extended the Watts–Strogatz model by explaining another important aspect of small world networks. He showed that the short paths not only exist but can be found using a simple greedy strategy with limited local information only. However, in our work we concentrate strictly on the basic properties of the Watts–Strogatz model and we leave these further improvements for future research.

Most well-known models of small world networks are stochastic. But deterministic models have the strong advantage that it is often possible to compute analytically their properties, for example, degree distribution, clustering coefficient, average path length, diameter, etc. Deterministic networks can be created by various techniques. We can modify regular graphs [6], or we can use standard graph operations such as the addition or the product of graphs [7], one can use recursive or iterative techniques based on the existence of cliques in a given network [8,19,20], and other mathematical methods.

In this paper, we focus on the small world network topology generated in a deterministic way, using the line graph operator. This deterministic approach enables one to obtain the relevant network parameters: degree distribution, clustering coefficient and diameter. We show that a network obtained in this way has strong clustering and a small diameter.

2. Definitions and notations

In this section we briefly introduce the important terms underlying our work and three axioms that must be satisfied by every Watts–Strogatz model of a small world network.

We consider only simple undirected connected graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We set $n = |V(G)|$ and $m = |E(G)|$. A *line graph* $L(G)$ has vertices corresponding to edges of G. That is, for every edge $e \in E(G)$ we have a vertex $v_e \in V(L(G))$. Two vertices of $L(G)$ are adjacent if and only if the corresponding edges in G share a common vertex. Denote $n' = |V(L(G))|$ and $m' = |E(L(G))|$. In the sequel we often use the fact that $n' = m$. We remark that the number of edges of $L(G)$ depends on the degree distribution in G.

The *diameter* of G is the greatest distance between two vertices in G:

$$\operatorname{diam}(G) = \max_{u,v \in V(G)} d(u, v). \tag{1}$$

Recall that the distance $d(u, v)$ is the number of edges in a shortest path starting at u and terminating at v. Regarding the diameter of line graphs, we will use the following statement given in [14]:

Theorem 1. *Let G be a connected graph with at least one edge. Then,*

$$\operatorname{diam}(G) - 1 \leqslant \operatorname{diam}\big(L(G)\big) \leqslant \operatorname{diam}(G) + 1.$$

A *clustering coefficient* is a measure of the degree to which vertices in a graph tend to cluster together and its value is always between 0 and 1. We can define a local and a global clustering coefficient. The (*local*) *clustering coefficient* of a vertex v of G, $\mathrm{CC}_G(v)$, is the ratio of the total number of existing connections between the neighbours $N_G(v)$ of v and the number of all possible connections between them. (Since G has no loops, $v \notin N_G(v)$.) We remark that if v has degree 0 or 1, then we set $\mathrm{CC}_G(v) = 0$. A (*global*) *clustering coefficient* can then be obtained by averaging the local clustering coefficients of all vertices of G, that is,

$$\mathrm{CC}(G) = \frac{1}{|V(G)|} \sum_{v \in V(G)} \mathrm{CC}_G(v). \tag{2}$$

There are two more definitions we need to include here. An edge is an (a, b)-edge if it has one endvertex of degree a and the other of degree b. An edge is *good* if it either has at least one endvertex of degree at least 3, or it lies in a triangle. Otherwise, it is a *bad edge*.

Now we state axioms for a graph G to be a Watts–Strogatz model for a small world network, see [9,16]:

(A1) *The graph G is sparse.* We require $|E(G)| \in O(n \lg n)$, that is, $|E(G)|/|V(G)| \in O(\lg |V(G)|)$.
(A2) *The diameter of G is small.* We require $\operatorname{diam}(G) \in O(\lg |V(G)|)$.
(A3) *The clustering coefficient* $\mathrm{CC}(G)$ *is large.* We require $\mathrm{CC}(G) \geqslant c$ for a positive constant c.

We remark that some authors prefer slightly different axioms. For example, they consider average distance instead of the diameter, or use the Θ notation instead of our O notation, etc. In what follows, we study sufficient conditions under which these axioms are satisfied by line graphs.

3. Line graph operator and axioms of small worlds

Here we study which of the properties (A1), (A2) and (A3) are preserved by the line graph operator. First, we consider the second axiom.

Proposition 2. *If G satisfies* (A2)*, then also $L(G)$ satisfies* (A2)*.*

Proof. Since the graph G is connected, $|V(L(G))| = m \geqslant n - 1$. By Theorem 1, $\operatorname{diam}(L(G)) \leqslant \operatorname{diam}(G) + 1$. Hence, $\operatorname{diam}(L(G)) \in O(\lg n) + 1 \subseteq O(\lg m)$. □

In order to prove an analogue of Proposition 2 for (A3), we first state two lemmas.

Lemma 3. *Let e be an edge in G. Then,*

(a) *If e is a bad edge, then $\mathrm{CC}_{L(G)}(v_e) = 0$;*
(b) *If e is a good edge, then $\mathrm{CC}_{L(G)}(v_e) \geqslant \frac{1}{3}$.*

Proof. Denote by u and v the endvertices of e. Further, denote by a (resp. b) the number of edges adjacent with e