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# Maximum weight independent sets in ($P_6$, co-banner)-free graphs

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ABSTRACT

The Maximum Weight Independent Set (MWIS) Problem on graphs with vertex weights asks for a set of pairwise nonadjacent vertices of maximum total weight. Being one of the most investigated problem on graphs, it is well known to be $\mathbb{NP}$-complete and hard to approximate. Several graph classes for which MWIS can be solved in polynomial time have been introduced in the literature. This note shows that MWIS can be solved in polynomial time for $(P_6, \text{co-banner})$-free graphs – where a $P_6$ is an induced path of 6 vertices and a co-banner is a graph with vertices $a, b, c, d, e$ and edges $ab, bc, cd, ce, de$ – so extending different analogous known results for other graph classes, namely, $P_4$-free, $2K_2$-free, $(P_5, \text{co-banner})$-free, and $(P_6, \text{triangle})$-free graphs. The solution algorithm is based on an idea/algorithm of Farber (1989) [10], leading to a dynamic programming approach for MWIS, and needs none of the aforementioned known results as sub-procedure.



## 1. Introduction

An *independent set* of a graph $G$ is a subset of pairwise nonadjacent vertices of $G$. The Maximum Independent Set (MIS) Problem asks for an independent vertex set of $G$ of maximum cardinality. If $w$ is a weight function on $V$ then the Maximum Weight Independent Set (MWIS) Problem asks for an independent set of maximum total weight. Being one of the most investigated problem on graphs, it is well known to be NP-complete ([GT20] in [13]) and hard to approximate.

Several graph classes for which MWIS can be solved in polynomial time have been introduced in the literature, such as e.g. $P_4$-free graphs [8], $2K_2$-free graphs [10], claw-free graphs (see [18] for the extension to fork-free graphs with related references).

This note shows that MWIS can be solved for ($P_6$, co-banner)-free in $O(mn^4)$ time by a simple algorithm – where a $P_6$ is an induced path of 6 vertices and a *co-banner* is a graph with vertices $a, b, c, d, e$ and edges $ab, bc, cd, ce, de$. That extends the following known results:

(a) the aforementioned results for $P_4$-free and for $2K_2$-free graphs;
(b) MWIS can be solved for $(P_5, \text{co-banner})$ in polynomial time (i.e., in $O(n^5)$ time); this result is based on the following structure property shown in [15]: if a prime graph contains an induced $2K_2$, then it contains an induced $P_5$ or one of two induced subgraphs each containing a co-banner (then MWIS can be finally reduced to the same problem for $2K_2$-free graphs; see also [7]);
(c) MWIS can be solved for $(P_6, \text{triangle})$-free graphs in polynomial time (i.e., in $O(n^2)$ time); this result is based on the following structural property shown in [5]: $(P_6, \text{triangle})$-free graphs have bounded clique-width and a corresponding clique-width expression can be constructed in $O(n^2)$ time (the result for MWIS applies to paw-free graphs too by [25]; see also [24]).

The computational complexity of M(W)IS is open for $P_6$-free graphs (see [4,20,22–24] for some subclasses of $P_6$-free graphs for which M(W)IS is polynomial) and is open even for $P_5$-free graphs. See Fig. 1. Actually $P_5$-free graphs constitute the unique minimal graph class defined by forbidding one single connected graph for which the computational complexity of M(W)IS is open (see [1]),

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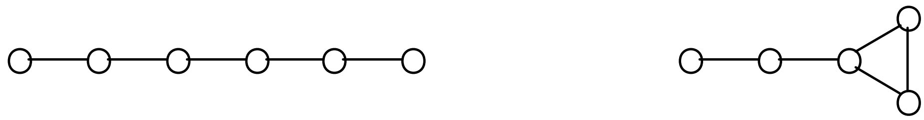

**Fig. 1.** A $P_6$, a co-banner.

although many structural properties on $P_5$-free and on $P_6$-free graphs have been introduced in the literature (see [2,3,16,17,21,26]).

### 1.1. Basic notation

For any missing notation or reference let us refer to [6].

Let $G$ be a graph and $V(G)$ be its vertex-set. For any vertex-set $U \subseteq V(G)$, let $N_G(U) = \{v \in V(G) \setminus (U \cup N(U))$: $v$ is adjacent to some $u \in U\}$ be the *neighborhood of* $U$ *in* $G$, and $A_G(U) = V \setminus (U \cup N(U))$ be the *anti-neighborhood of* $U$ *in* $G$. If $U = \{u_1, \dots, u_k\}$, then let us simply write $N_G(u_1, \dots, u_k)$ instead of $N_G(U)$, and $A_G(u_1, \dots, u_k)$ instead of $A_G(U)$.

For any subset $U \subseteq V(G)$ let $G[U]$ be the subgraph of $G$ induced by $U$: for a fluid description, we often denote $G[U]$ simply as $U$ when that does not generate ambiguity. A *component* of $G$ is a maximal connected subgraph of $G$. A component of $G$ is *trivial* if it is an isolated vertex, and is *nontrivial* otherwise. For a vertex $v \in V(G)$ and for a subset $U \subset V(G)$ (with $v \notin U$), let us say that $v$ *sees* $U$ if $v$ is adjacent to some vertex of $U$; $v$ *dominates* $U$ if $v$ is adjacent to each vertex of $U$.

For any graph $H$, let us say that $G$ is *$H$-free* if $G$ has no induced subgraph isomorphic to $H$.

Graph $G$ is *prime* if for any $U \subset V(G)$ with $|U| \geqslant 2$ there is a vertex $v \in V(G) \setminus U$ such that $v$ is adjacent to some vertex of $U$ and is nonadjacent to some vertex of $U$.

### 1.2. Preliminary

The solution to solve MWIS for $(P_6, \text{co-banner})$-free graphs is based on the idea introduced by Farber to prove that every $2K_2$-free graph has $O(n^2)$ maximal independent sets [10]. The proof directly gave a polynomial time algorithm to solve MWIS for $2K_2$-free graphs by a dynamic programming approach to the problem, which has been the basis for extensions of the result (see e.g. [11,19]).

In that context two tools are used.

The first one is the following well-known result, reported in [18] as Theorem 1 with an explicit proof, which allows to focus just on prime $(P_6, \text{co-banner})$-free graphs.

**Theorem 1.** *Let $\mathcal{C}$ be a hereditary class of graphs (i.e., defined by forbidding induced subgraphs). If the MWIS problem can be solved in time $T$ for every prime graph in $\mathcal{C}$, then MWIS can be solved in time $T$ for every graph in $\mathcal{C}$.*

The second one is the following easy but useful result.

**Lemma 1.** *Let $G$ be a prime $(P_6, \text{co-banner})$-free graph. Then the anti-neighborhood of each edge of $G$ induces a bipartite graph.*

**Proof.** Let $xy$ be an edge of $G$. If $A_G(x, y)$ has no nontrivial components, then $A_G(x, y)$ is (trivially) bipartite. Then let $K$ be a nontrivial component of $A_G(x, y)$. Since $G$ is prime (then connected) there exists a vertex $w \in N_G(x) \cup N_G(y)$ distinguishing $K$, i.e., there is an edge $ab$ of $K$ such that $w$ is adjacent to $a$ and is nonadjacent to $b$. To avoid that $x, y, w, a, b$ induce a co-banner, $w \notin N_G(x) \cap N_G(y)$. Then assume without loss of generality that $w \in N_G(x) \setminus N_G(y)$. Let us write $K' = K \cap N_G(w)$ and $K'' = K \setminus N(w)$ (then $\{K', K''\}$ is a partition of $K$).

Note that $K'$ is an independent set, otherwise a co-banner arises involving $y, x, w$. Let us show that $K''$ is an independent set as well. By contradiction assume that there is an edge $cd$ in $K''$. Since $K$ is connected, one can choose $cd$ in order that there is a vertex $k \in K'$ adjacent to at least one from $\{c, d\}$, say to $c$ without loss of generality. If $k$ is adjacent to $d$, then $x, w, k, c, d$ induce a co-banner (contradiction). If $k$ is nonadjacent to $d$, then $y, x, w, k, c, d$ induce a $P_6$ (contradiction). Then $K''$ is an independent set. Then $K$ induces a bipartite graph, and the lemma follows. $\square$

To conclude let us point out that, as shown later, MWIS for $(P_6, \text{co-banner})$-free graphs can be finally reduced to the same problem for bipartite ($P_6$-free) graphs. It is well known that MWIS is solvable for bipartite graphs in polynomial time (i.e. in $O(n^{2.5})$ time) [9]. However to solve MWIS for bipartite $P_6$-free graphs, let us refer to the following result [12] (based on the bounded clique-width property, see also [14]) which is better in terms of computational complexity.

**Theorem 2.** *(See [12].) The MWIS problem for bipartite $P_6$-free graphs can be solved in $O(n + m)$ time.*

## 2. Maximum weight independent sets in $(P_6, \text{co-banner})$-free graphs

In this section let us show that MWIS can be solved for $(P_6$, co-banner$)$-free graphs in $O(n^4 m)$ time. Throughout this section let $G$ be a prime $(P_6, \text{co-banner})$-free graph, with $V(G) = \{v_1, v_2, \dots, v_n\}$. Let us denote by $G_i$ the subgraph of $G$ induced by vertices $v_1, v_2, \dots, v_i$.

The solution consists of two phases.

In Phase 1, one generates a family $S$ of polynomially many subsets of $V(G)$ through Algorithm Alpha (described below) in polynomial time: the assumption that $G$ is prime $(P_6, \text{co-banner})$-free warrants that every maximal independent set of $G$ is contained in at least one member of $S$ and that each member of $S$ is an independent set or induces a bipartite graph.

In Phase 2, one solves MWIS for the subgraph induced by each member of $S$ and choose a best solution: that can