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Partitioning extended P_4-laden graphs into cliques and stable sets

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ABSTRACT

A (k, ℓ)-cocoloring of a graph is a partition of its vertex set into at most k stable sets and at most ℓ cliques. It is known that deciding if a graph is (k, ℓ)-cocolorable is NP-complete. A graph is extended P_4-laden if every induced subgraph with at most six vertices that contains more than two induced P_4's is $\{2K_2, C_4\}$-free. Extended P_4-laden graphs generalize cographs, P_4-sparse and P_4-tidy graphs. In this paper, we obtain a linear time algorithm to decide if, given $k, \ell \geqslant 0$, an extended P_4-laden graph is (k, ℓ)-cocolorable. Consequently, we obtain a polynomial time algorithm to determine the cochromatic number and the split chromatic number of an extended P_4-laden graph. Finally, we present a polynomial time algorithm to find a maximum induced (k, ℓ)-cocolorable subgraph of an extended P_4-laden graph, generalizing the main results of Bravo et al. (2011) [4] and Demange et al. (2005) [5].

1. Introduction

Many hard problems on graphs, such as the coloring problem or the vertex cover problem, are vertex partition problems. In general, the objective of a partition problem is to partition the vertex set of a graph into disjoint subsets (or classes) which satisfy certain conditions. Such conditions can be *internal*, as in the graph coloring problem, where each class must be a stable set, or *external*, as in the acyclic coloring problem, where every two classes induce an acyclic subgraph. An interesting partition problem is the partition of the vertex set into cliques and stable sets.

We say that a graph is (k, ℓ)-cocolorable (also referred in the literature as (k, ℓ)-colorable) if its vertex set can be partitioned in at most k stable sets and at most ℓ cliques. This concept was first introduced in [16] and attracted the attention of Erdős [7–9] as a natural extension of the graph coloring problem. As an example, split graphs are the $(1, 1)$-cocolorable graphs. The cochromatic number of a graph G is the smallest integer $z(G)$ such that G is (k, ℓ)-cocolorable for $k + \ell = z(G)$ [13].

An interesting application of the cochromatic number is to partition a permutation into increasing or decreasing subsequences. Unfortunately, finding the cochromatic number is NP-hard even for permutation graphs [18]. In 2010, Heggernes et al. obtained a fixed-parameter algorithm to solve this problem for permutation graphs (and for perfect graphs) [14].

In [1,2], Brandstädt obtained a polynomial time algorithm to decide if a graph is $(2, 2)$-cocolorable and proved that deciding if a graph is (k, ℓ)-cocolorable is NP-Complete for $k \geqslant 3$ or $\ell \geqslant 3$. Recently, polynomial time algorithms was obtained for chordal graphs [15], cographs [3,5,10] and P_4-sparse graphs [4]. This problem was also investigated for perfect graphs in [11].

In this paper, we investigate the cocoloring problem for extended P_4-laden graphs, which are the graphs such that every induced subgraph with at most six vertices contains at most two induced P_4's or is $\{2K_2, C_4\}$-free. This graph class was introduced in [12], and a motivation to develop algorithms for extended P_4-laden graphs lies on the fact that they are on the top of a widely studied hierarchy of

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classes containing many graphs with few P_4's, including cographs, P_4-sparse, P_4-lite, P_4-laden and P_4-tidy graphs. Therefore, solving interesting problems in an efficient way for extended P_4-laden graphs immediately imply efficient, generalized algorithms for all these classes. Another motivation is that extended P_4-laden graphs are not contained in perfect graphs; hence this work brings good examples of coloring applications not specifically related to perfection.

In this paper, we present a linear time algorithm to decide if an extended P_4-laden graph is (k,ℓ)-cocolorable for $k,\ell \geqslant 0$. Consequently, we obtain a polynomial time algorithm to determine the cochromatic number of an extended P_4-laden graph. Our result also implies a polynomial time algorithm to determine the *split chromatic number* of an extended P_4-laden graph G, which is defined as the minimum $p = \max\{k,\ell\}$ such that G is (k,ℓ)-cocolorable (see [5,6]). Finally, we present a polynomial time algorithm to find a maximum induced (k,ℓ)-cocolorable subgraph of an extended P_4-laden graph.

In what follows, we denote by K_n a complete graph with n vertices. An induced path with n vertices is denoted by P_n. An induced cycle with n vertices is denoted by C_n. A graph G is *empty* if $V(G)=\emptyset$. In general, we assume for a graph G that $|V(G)|=n$ and $|E(G)|=m$.

2. Extended P_4-laden graphs

Using modular decomposition, Giakoumakis [12] proved an important structural characterization for extended P_4-laden graphs by special graphs, called here *pseudo-splits* and *quasi-spiders*.

Given a split graph G with vertex set partition $(\mathcal{S},\mathcal{C})$, where $\mathcal{S}$ is a stable set and $\mathcal{C}$ is a clique, we say that G is *original* if every vertex in $\mathcal{S}$ has a non-neighbor in $\mathcal{C}$ and every vertex in $\mathcal{C}$ has a neighbor in $\mathcal{S}$. We say that a graph G is a *pseudo-split* if its vertex set has a partition $(\mathcal{S},\mathcal{C},\mathcal{R})$ such that $\mathcal{S}$ induces a stable set, $\mathcal{C}$ induces a clique, $\mathcal{S}\cup\mathcal{C}$ induces an original split graph (where $\mathcal{S}\cup\mathcal{C}\neq\emptyset$) and every vertex of $\mathcal{R}$ is adjacent to every vertex of $\mathcal{C}$ and non-adjacent to every vertex of $\mathcal{S}$.

We can see $\mathcal{S}$, $\mathcal{C}$ and $\mathcal{R}$ respectively as the legs, the body and the head of the pseudo-split. Observe that $\mathcal{R}$ can be empty and, in this case, we say that the pseudo-split is *headless*. Also notice that the complement of a pseudo-split is also a pseudo-split.

We say that a pseudo-split G is a *spider* $(\mathcal{S},\mathcal{C},\mathcal{R})$ if $|\mathcal{S}|=|\mathcal{C}|=k$, $\mathcal{S}=\{s_1,\dots,s_k\}$ and $\mathcal{C}=\{c_1,\dots,c_k\}$, where

(a) s_i is adjacent to c_j if and only if $i=j$, for every $1\leqslant i,j\leqslant k$ (*thin spider*); or
(b) s_i is adjacent to c_j if and only if $i\neq j$, for every $1\leqslant i,j\leqslant k$ (*thick spider*).

Notice that the complement of a thin spider is a thick spider, and vice-versa.

A *quasi-spider* is a graph obtained by a spider $(\mathcal{S},\mathcal{C},\mathcal{R})$ with at most one vertex from $\mathcal{S}\cup\mathcal{C}$ replaced by K_2 or $\overline{K_2}$ (keeping the neighborhood). Clearly, every spider is a quasi-spider.

The *disjoint union* (or simply *union*) of two graphs G_1 and G_2 is the graph $G_1\cup G_2$, where $V(G_1\cup G_2)=V(G_1)\cup V(G_2)$ and $E(G_1\cup G_2)=E(G_1)\cup E(G_2)$. The *join* of two graphs G_1 and G_2 is the graph G_1+G_2, where $V(G_1+G_2)=V(G_1)\cup V(G_2)$ and $E(G_1+G_2)=E(G_1)\cup E(G_2)\cup\{xy \mid x\in V(G_1),\, y\in V(G_2)\}$.

Theorem 2.1. *(See [12].) A graph G is extended P_4-laden if and only if exactly one of the following conditions is satisfied:*

(a) *G is the disjoint union of two non-empty extended P_4-laden graphs.*
(b) *G is the join of two non-empty extended P_4-laden graphs.*
(c) *G is a quasi-spider such that its head induces an extended P_4-laden graph.*
(d) *G is a pseudo-split such that its head induces an extended P_4-laden graph.*
(e) *G is isomorphic to C_5, P_5 or $\overline{P_5}$.*
(f) *G has only one vertex or $V(G)=\emptyset$.*

This theorem suggests a natural decomposition for extended P_4-laden graphs. Let T_G be the tree decomposition of an extended P_4-laden graph G. Every node u of T_G represents an induced subgraph $G(u)$ of G. The root r of T_G represents the original graph $G=G(r)$. The leaves are only C_5, P_5, $\overline{P_5}$, pseudo-splits without head, quasi-spiders without head or a single vertex; in addition, leaves constitute a partition of $V(G)$. The non-leaf nodes of T_G are called *internal*. In case (a) (resp. (b)) of the above theorem, an internal node is the disjoint union (resp. join) of its children. In case (c) (resp. (d)) of the above theorem, if $R\neq\emptyset$, an internal node with have $\mathcal{S}\cup\mathcal{C}$ and $\mathcal{R}$ as its children. From [12], the tree T_G is unique up to isomorphism and can be obtained in $O(n+m)$ time.

3. (k,ℓ)-cocoloring extended P_4-laden graphs for $k,\ell\geqslant 2$

In this section, we present a polynomial time algorithm to recognize (k,ℓ)-cocolorable extended P_4-laden graphs for $k,\ell\geqslant 2$. The key idea is, given an extended P_4-laden graph G, to obtain a cograph G^* by removing all induced P_4's from G. We prove that, if $k,\ell\geqslant 2$, G is (k,ℓ)-cocolorable if and only if G^* is (k,ℓ)-cocolorable. Since recognizing (k,ℓ)-cocolorable cographs is solvable in polynomial time (and even in $O(n)$ time if k,ℓ are fixed), we are done (see [3,5]).

Clearly, a graph G is extended P_4-laden if and only if $\overline{G}$ is extended P_4-laden, and a graph G is (k,ℓ)-cocolorable if and only if $\overline{G}$ is (ℓ,k)-cocolorable.

Let G be an extended P_4-laden graph. Let T_G be the decomposition tree of G. To obtain the cograph G^*, we will apply the rules below for each node u of T_G, where $G(u)$ is the induced subgraph of G represented by u. To ease the description of the rules, consider graph P_5 with edge set $\{ab,bc,cd,de\}$, graph $\overline{P_5}$ with edge set $\{ac,ad,ae,bd,be,ce\}$, and graph C_5 with edge set $\{ab,bc,cd,de,ae\}$.

Rule 1: If $G(u)$ is isomorphic to C_5: Add edge ac, remove edges ae,cd, add a new vertex f adjacent to e.
Rule 2: If $G(u)$ is isomorphic to P_5: Remove edge bc.
Rule 3: If $G(u)$ is isomorphic to $\overline{P_5}$: Add edge bc.