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Satisfiability problem for modal logic with global counting operators coded in binary is NEXPTIME-complete

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ABSTRACT

This paper provides a proof of NEXPTIME-completeness of the satisfiability problem for the logic $\mathsf{K}(\mathsf{E}_n)$ (modal logic K with global counting operators), where number constraints are coded in binary. Hitherto the tight complexity bounds (namely EXPTIME-completeness) have been established only for this logic with number restrictions coded in unary. The upper bound is established by showing that $\mathsf{K}(\mathsf{E}_n)$ has the exponential-size model property and the lower bound follows from reducibility of exponential bounded tiling problem to $\mathsf{K}(\mathsf{E}_n)$.

1. Introduction

Counting modalities were first introduced by Fine in [1] under the name of *graded modalities*. They allowed expressing a number of successors of a particular world, at which a certain formula holds. In particular, a formula $\Diamond_{=n}\top$ expresses the fact that the current world has exactly n successors. In [2] a filtration-based proof of decidability of several graded modal logics is provided. However, no complexity results are presented. A first systematic treatment of the complexity of various graded modal logics, for both unary and binary coding of numerical subscripts, can be found in [3].

In [4] Areces et al. recalled modal logics with counting operators ($\mathcal{MLC}$). In these logics global counting operators $\mathsf{E}_{>n}$, $\mathsf{E}_{<n}$ and $\mathsf{E}_{=n}$ were added to a modal language with the ordinary modalities. Global counting operators increase the expressive power of a logic by allowing the definition of nominals, the universal modality, and counting the cardinality of a domain (by a formula $\mathsf{E}_{=n}\top$). It also enables the formalisation of natural language queries that involve numbers.

In terms of computational complexity, tight bounds were established for $\mathcal{MLC}$ with number constraints coded in unary. In particular, [5] states and [4] recalls EXPTIME-completeness. However, the bounds for $\mathcal{MLC}$ with number constraints coded in binary have so far remained loose. In [4] EXPTIME-hardness and membership in 2NEXPTIME is recalled, which leaves room for a tight result.

In this paper we prove NEXPTIME-completeness for the logic with number constraints coded in binary. In Section 2 we provide a characterisation of $\mathcal{MLC}$, which we present under the name $\mathsf{K}(\mathsf{E}_\mathsf{n})$. In Section 3 we establish an upper bound by proving that $\mathsf{K}(\mathsf{E}_\mathsf{n})$ has the *exponential-size model property*. Section 4 is devoted to the reduction of the exponential *bounded tiling problem* to $\mathsf{K}(\mathsf{E}_\mathsf{n})$, which results in NEXPTIME-hardness of the logic. A brief summary of the paper and concluding remarks are provided in Section 5.

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2. The logic $\mathsf{K}(\mathsf{E_n})$

First, we define the language of $\mathsf{K}(\mathsf{E_n})$. Let PROP $= \{p_1, p_2, \ldots\}$ be a countable set of propositional letters. We define a set FORM of formulas of $\mathsf{K}(\mathsf{E_n})$ as follows, where $p \in$ PROP, $\varphi, \psi \in$ FORM, $n \in \mathbb{N}$:

$$\text{FORM} ::= \top \mid p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Diamond\varphi \mid \mathsf{E}_{>n}\varphi.$$

Other Boolean operators and the $\Box$ operator are defined in a standard way. Moreover, we introduce two additional counting operators $\mathsf{E}_{<n}$ and $\mathsf{E}_{=n}$:

$$\mathsf{E}_{<n+1}\varphi := \neg\mathsf{E}_{>n}\varphi,$$

$$\mathsf{E}_{=n+1}\varphi := \mathsf{E}_{>n}\varphi \wedge \neg\mathsf{E}_{>n+1}\varphi,$$

$$\mathsf{E}_{=0}\varphi := \neg\mathsf{E}_{>0}\varphi.$$

The logic $\mathsf{K}(\mathsf{E_n})$ allows encoding the universal modality A as: $\mathsf{A}\varphi := \mathsf{E}_{=0}\neg\varphi$.

A *model* for $\mathsf{K}(\mathsf{E_n})$ is a triple $\langle W, R, V\rangle$, where W is a non-empty set whose elements are usually called worlds, R is a binary relation on W, and $V :$ PROP $\rightarrow \mathcal{P}(W)$ is a valuation function assigning to each $p \in$ PROP a set of worlds in which p holds. Given a model $\langle W, R, V\rangle$ and $w \in W$, the semantics of $\mathsf{K}(\mathsf{E_n})$ is defined as follows:

$\mathfrak{M}, w \models p$	iff	$w \in V(p)$, $p \in$ PROP,
$\mathfrak{M}, w \models \neg\varphi$	iff	$\mathfrak{M}, w \not\models \varphi$,
$\mathfrak{M}, w \models \varphi \wedge \psi$	iff	$\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$
$\mathfrak{M}, w \models \Diamond\varphi$	iff	there exists $v \in W$ such that wRv and $\mathfrak{M}, v \models \varphi$,
$\mathfrak{M}, w \models \mathsf{E}_{>n}\varphi$	iff	$\mathsf{Card}(\{v \mid \mathfrak{M}, v \models \varphi\}) > n$,

where $\mathsf{Card}(A)$ denotes the cardinality of the set A.

3. Membership in NExpTime

Membership in NExpTime is shown by proving the exponential-size model property. Membership in NExpTime also follows from the existence of a tableau algorithm running in NExpTime [6] or the existence of a standard translation from $\mathsf{K}(\mathsf{E_n})$ to $\mathcal{C}^2$ (the two-variable fragment of first order logic with counting quantifiers) where number constraints are coded in binary, which was proven to be NExpTime-complete [7]. An advantage of providing a direct proof of the finite model property is obtaining precise bounds of the sizes of models.

Lemma 1 *(Finite Model Property). Let φ be any $\mathsf{K}(\mathsf{E_n})$ formula. If φ has a satisfying model, it also has a satisfying model of the size not exceeding $2^{\mathsf{Card}(\mathsf{Sub}(\varphi))} \cdot (n+1)$, where $\mathsf{Sub}(\varphi)$ is the set of all subformulae of φ and $n = \max\{m \colon \mathsf{E}_{>m}\psi \in \mathsf{Sub}(\varphi)\}$.*

Proof. Let φ be a formula satisfiable on a (possibly infinite) model $\mathfrak{M} = \langle W, R, V\rangle$. We show that there exists a finite model $\mathfrak{M}' = \langle W', R', V'\rangle$ on which φ is satisfiable.

We proceed in two steps. In the first step we exploit a filtration-like method to divide the universe W into a finite number of equivalence classes. We fix the equivalence relation $\leftrightsquigarrow_{\mathsf{Sub}(\varphi)}$ in the following way: $w \leftrightsquigarrow_{\mathsf{Sub}(\varphi)} v$ iff for all $\psi \in \mathsf{Sub}(\varphi)$ we have $\mathfrak{M}, w \models \psi$ iff $\mathfrak{M}, v \models \psi$. It is straightforward that there are only finitely many such equivalence classes, namely $2^{\mathsf{Card}(\mathsf{Sub}(\varphi))}$ many.

In the second step we abandon the ordinary filtration procedure. Instead of merging all worlds from the equivalence classes, we reduce the cardinality of each class in the following manner. Let $[w] \subseteq W$ be an arbitrary $\leftrightsquigarrow_{\mathsf{Sub}(\varphi)}$-equivalence class. If $\mathsf{Card}([w]) > n+1$ then we delete all but $n+1$ arbitrary worlds from $[w]$. If $\mathsf{Card}([w]) \leqslant n+1$ then we leave $[w]$ unchanged. Henceforth, we denote such a reduct of $[w]$ as $[w]'$. Next, from each reduced equivalence class $[w]'$ we pick an arbitrary representative w_0. We set a new model $\mathfrak{M}' = \langle W', R', V'\rangle$, where $W' = \bigcup\{[w]' \mid [w] \in W/_{\leftrightsquigarrow_{\mathsf{Sub}(\varphi)}}\}$, $R' = R \upharpoonright W' \cup \bigcup_{[w],[v] \in W/_{\leftrightsquigarrow_{\mathsf{Sub}(\varphi)}}} \{(w, v_0) \mid w \in [w]'$ and there exists $v \in [v] \setminus [v]'$ such that $R(w, v)$ and $v_0 \in [v]'\}$, and $V' = V \upharpoonright W'$.

The proof that $\mathfrak{M}'$ is a model for φ is by induction on the complexity of the elements of $\mathsf{Sub}(\varphi)$. The Boolean cases are obvious and follow directly from the definition of W' and V'.

The $\Diamond$ case is proven in the following way. Suppose that a formula $\Diamond\psi$ is satisfiable on $\mathfrak{M}$. It means that there exists a $w \in W$ such that $\mathfrak{M}, w \models \Diamond\psi$. We pick an arbitrary $\hat{w} \in [w]'$. By definition of $\leftrightsquigarrow_{\mathsf{Sub}(\varphi)}$ it follows that $\mathfrak{M}, \hat{w} \models \Diamond\psi$. Consequently, we can find $v \in W$ such that $(\hat{w}, v) \in R$ and $\mathfrak{M}, v \models \psi$. If $v \in [v]'$ then we also have $\mathfrak{M}', \hat{w} \models \Diamond\psi$. Otherwise, by definition of $\leftrightsquigarrow_{\mathsf{Sub}(\varphi)}$ and R' there exists $v_0 \in [v]'$ such that $(\hat{w}, v_0) \in R'$ and $\mathfrak{M}, v_0 \models \psi$. Therefore, $\Diamond\psi$ is satisfiable on $\mathfrak{M}$.

Now, assume that a formula $\mathsf{E}_{>m}\psi$ is satisfied by $\mathfrak{M}$. It means that there exist more than m worlds in which ψ holds. Two cases may occur. Either ψ holds in elements of (at least one) equivalence class $[w]$ such that $\mathsf{Card}([w]) > n$. Then, by the construction of $[w]'$, we obtain that $\mathsf{E}_{>m}\psi$ is satisfied by $\mathfrak{M}'$. Otherwise ψ holds in elements of the equivalence classes $[w_{i_1}], \ldots, [w_{i_k}]$ such that $\mathsf{Card}([w_{i_j}]) \leqslant n$ and $\sum_{j=1}^{k} \mathsf{Card}([w_{i_j}]) > n$. But by construction of $[w_{i_j}]'$ these classes remained unchanged in W', therefore $\sum_{j=1}^{k} \mathsf{Card}([w_{i_j}]') > n$. It follows that $\mathsf{E}_{>m}\psi$ is satisfied by $\mathfrak{M}'$. The reduction of the size of W cannot disturb satisfiability of the formulas $\mathsf{E}_{<m}\psi$ on $\mathfrak{M}'$.

Since $\mathsf{Card}(W/_{\leftrightsquigarrow_{\mathsf{Sub}(\varphi)}}) = 2^{\mathsf{Card}(\mathsf{Sub}(\varphi))}$ and for each $[w]'$ obtained from $[w] \in W/_{\leftrightsquigarrow_{\mathsf{Sub}(\varphi)}} \mathsf{Card}([w]') \leqslant n+1$, it is clear that $\mathsf{Card}(W') \leqslant 2^{\mathsf{Card}(\mathsf{Sub}(\varphi))} \cdot (n+1)$. This completes the proof. $\Box$

We can conclude:

Theorem 2. *(See [7,6].) The satisfiability problem for modal the logic $\mathsf{K}(\mathsf{E_n})$ with number constraints coded in binary is in NExpTime.*

4. NExpTime-hardness

NExpTime-hardness is proven by reducing a standard exponential *bounded tiling problem* [8].

By a tile type T we understand a quadruple of colours $(\mathsf{left}_T, \mathsf{up}_T, \mathsf{right}_T, \mathsf{down}_T)$. Given a finite set of tile-types $\mathsf{T} = \{T_0, \ldots, T_m\}$ and a finite square grid $k \times k$, the bounded