Information Processing Letters 112 (2012) 911–915

Contents lists available at SciVerse ScienceDirect

Information Processing Letters

www.elsevier.com/locate/ipl

An improved analysis of SRPT scheduling algorithm on the basis of functional optimization

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ARTICLE INFO

Article history:
Received 2 April 2012
Received in revised form 28 June 2012
Accepted 17 August 2012
Available online 20 August 2012
Communicated by M. Yamashita

Keywords:
Analysis of algorithms
On-line algorithms
Scheduling
Competitive analysis
Functional analysis

ABSTRACT

The competitive performance of the SRPT scheduling algorithm has been open for a long time except for being 2-competitive, where the objective is to minimize the total completion time. Chung et al. proved that the SRPT algorithm is 1.857-competitive. In this paper we improve their analysis and show a 1.792-competitiveness. We clearly mention that our result is *not* the best so far, since Sitters recently proved the algorithm is 1.250-competitive. Nevertheless, it is still well worth reporting our analytical method; our analysis is based on the *modern functional optimization*, which can scarcely be found in the literature on the analysis of algorithms. Our aim is to illustrate the potentiality of functional optimization with a concrete application.

1. Introduction

The SRPT algorithm is a simple algorithm for online job scheduling to minimize the total completion time. Despite its simplicity, the theoretical performance has been open for a long time, except for being 2-competitive [4]. Breaking the barrier of 2, Chung et al. [1] proved that the SRPT algorithm is 1.857-competitive. The proof was done by using a probabilistic method in which the choice of a probabilistic distribution influences the resulting competitiveness. In this paper we obtain an optimal distribution and consequently show a 1.792-competitiveness. More precisely, we formulate a linear program over a function space and find a solution which satisfies the optimality condition.

We clearly mention that our result is *not* the best so far; Sitters [7] recently proved by a quite different argument that the SRPT algorithm is 1.250-competitive. Nevertheless, it is still well worth reporting our analytical method. Although the analysis of algorithms by means of functional optimization is of great potentiality, its application can scarcely be found in the literature.

Optimization theory has played a significant role in the design and analysis of algorithms. Polynomial-time algorithms for LP or SDP are often embedded as a subroutine. Duality theory is comprehensively applied to performance analysis. In most of the researches, however, the problem to be solved is an optimization over finite-dimensional vector space. To the best of our knowledge, merely the work [2] dealt with a function space, which is regarded as an infinite-dimensional vector space, as we do in this paper.

2. Preliminaries for the functional analysis

We assume all integrals appearing in this paper to be the Lebesgue integrals. $L^1[0,1]$ denotes the set of Lebesgue measurable functions $f:[0,1]\to\mathbb{R}$ for which $\int_0^1 |f(x)|\,dx<\infty$. $L^\infty[0,1]$ stands for the set of Lebesgue measurable functions $f:[0,1]\to\mathbb{R}$ whose image is bounded except on a set of measure zero. Indeed, we will only handle functions whose image is simply bounded. We denote by $C[0,1]$ and $\mathrm{ND}[0,1]$ the set of continuous and nondecreasing functions on $[0,1]$, respectively.

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http://dx.doi.org/10.1016/j.ipl.2012.08.009

We also employ the Lebesgue–Stieltjes integral for dealing with a probabilistic distribution which may not have a density function. What should be noted is just the treatment of a discontinuous point: Let $F \in \mathrm{ND}[0,1]$ that is discontinuous at $c \in [0,1]$ and differentiable elsewhere, and $g \in C[0,1]$. Denoting the derivative of F by f, we calculate

$$\int_0^1 g(x)\,dF(x) = \int_0^c g(x) f(x)\,dx + \int_c^1 g(x) f(x)\,dx + g(c)\big(F(c+) - F(c-)\big),$$

where $F(c+)$ and $F(c-)$ stand for $\lim_{\epsilon \to +0} F(c+\epsilon)$ and $\lim_{\epsilon \to +0} F(c-\epsilon)$, respectively.

The purpose of the Lebesgue integral and the above function spaces is to rigorously specify a space to which a dual variable in an optimization problem belongs. It is well known that dual variables that correspond to a finite set of constraints form $\mathbb{R}^n$. Concerning a constraint that holds true over a real interval, however, a more careful and involved argument is required. Although definition of such space is rather intricate, the resulting functions in this paper are all elementary. Therefore, any integral of a function given in the explicit form may be considered as the Riemann integral. As for the Lebesgue integral and function spaces, refer to [5,3,6].

3. Formulation and observation

Chung et al. [1] studied the performance of the SRPT algorithm for online preemptive job scheduling on identical parallel machines, where the objective is to minimize the total completion time. Here, the SRPT algorithm always executes the jobs with shortest remaining processing time. Although the SRPT algorithm schedules deterministically, the following lemma evaluates the competitive performance based on a probabilistic argument.

Lemma 1. *(See [1].) Let X be a random variable on the interval $[0,1]$. Then the competitive ratio of SRPT is at most*

$$E[X] + \max_{0 \leqslant a \leqslant 1} \frac{1}{1+a}\big(\Pr[0 \leqslant X \leqslant a] + \Pr[a < X \leqslant 1] E[X \mid a < X \leqslant 1]\big) + 1. \tag{1}$$

Chung et al. applied the specific cumulative distribution $F(x) = \Pr[X \leqslant x] = 1 - (1-x)^7$.

Theorem 1. *(See [1].) The SRPT algorithm is 1.857-competitive.*

We find an optimal distribution $F_0(x) = 1 - \frac{1}{1+\gamma_0} \ln \frac{1-x}{1-\gamma_0}$ for $0 \leqslant x \leqslant \gamma_0$ and 1 for $\gamma_0 < x \leqslant 1$ by the following argument, and obtain the next theorem, where $\gamma_0 \approx 0.442$.

Theorem 2. *The SRPT algorithm is 1.792-competitive, which is the best possible on the basis of Lemma 1.*

As we have mentioned before, our result is *not* the best so far.

Theorem 3. *(See [7].) The SRPT algorithm is 1.250-competitive.*

We begin with formulating a functional optimization problem to find a distribution which minimizes (1). The set of cumulative distributions $F(x) = \Pr[X \leqslant x]$ on $[0,1]$ can be identified with $\mathrm{ND}[0,1]$. For $F \in \mathrm{ND}[0,1]$, we define

$$J(F) = E[X] = \int_0^1 x\,dF(x);$$

$$A(F) = \Pr[0 \leqslant X \leqslant 1] = \int_0^1 dF(x);$$

$$B(F)(a) = \frac{1}{1+a}\big(\Pr[0 \leqslant X \leqslant a] + \Pr[a < X \leqslant 1] E[X \mid a < X \leqslant 1]\big) = \frac{1}{1+a}\left(\int_0^a dF(x) + \int_a^1 x\,dF(x)\right), \quad a \in [0,1].$$

Then the problem is a *linear functional optimization problem*:

$$\begin{aligned} (\mathcal{P}) \quad & \text{minimize} \quad J(F) + \beta \\ & \text{subject to} \quad A(F) = 1, && (2) \\ & \qquad B(F)(a) \leqslant \beta, \quad a \in [0,1], && (3) \\ & \qquad F \in \mathrm{ND}[0,1]. && (4) \end{aligned}$$

Unlike classical functional problems that can be solved by the calculus of variations, our problem involves optimization with the modern functional analysis since it includes the inequality constraint (3) over a real interval.

It is easily seen that $J : \mathrm{ND}[0,1] \to \mathbb{R}$, $A : \mathrm{ND}[0,1] \to \mathbb{R}$ are linear operators. Here we should specify the whole set of $B(F)$. We first determine the image space of the linear operator B. The mapping $a \mapsto B(F)(a)$ is measurable, since the mappings $a \mapsto \int_0^a dF(x)$, $a \mapsto \int_a^1 x\,dF(x)$, and $a \mapsto \frac{1}{1+a}$ are monotonic and therefore measurable. In addition, we have

$$\big|B(F)(a)\big| \leqslant \frac{1}{1+a}\left(\left|\int_0^a dF(x)\right| + \left|\int_a^1 x\,dF(x)\right|\right) \leqslant \frac{1}{1+a}\left(\int_0^1 dF(x) + \int_0^1 x\,dF(x)\right).$$

Thus $B(F) \in L^\infty[0,1]$ for each $F \in \mathrm{ND}[0,1]$.

The rest of this paper is dedicated to solving $(\mathcal{P})$. The sketch is as follows. First, we derive a sufficient condition for optimality. Unfortunately, it is difficult to find a solution which satisfies the condition in a straightforward manner. Thus, we next formulate a subproblem $(\mathcal{Q})$ over a narrowed solution space, with the help of the guess of a solution. $(\mathcal{Q})$ is solved analytically. Finally, we confirm that the obtained solution is also optimal for $(\mathcal{P})$.