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A K-trivial set which is not jump traceable at certain orders

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ABSTRACT

We construct a K-trivial c.e. set which is not jump traceable at any order in $o(\log x)$.

1. Introduction

The study of algorithmic randomness, specifically Martin-Löf randomness, has precipitated the study of those sets which are low for Martin-Löf randomness. This class was shown by Nies and his co-authors [7] to coincide with a number of other classes of computably weak sets, including the low for K sets and the K-trivial sets (for brevity, we will use the name K-trivial). For background on these notions, see the texts by Nies [8] or Downey and Hirschfeldt [3].

Motivated by the robust nature of this class, lowness for other randomness notions has been studied. Terwijn and Zambella [9] and then Kjos-Hanssen, Nies and Stephan [5] characterized those sets which are low for Schnorr randomness, while Bienvenu, Downey, Greenberg, Nies and Turetsky [1] characterized those sets which are low for Demuth randomness. In both cases, a notion of traceability was used to give a purely combinatorial characterization of the class. This stands in contrast to the K-trivial sets; although many different characterizations are known, they all involve effective randomness or measure in some fashion.

A (partial) function $f : \omega \to \omega$ is *traced* by a sequence of sets $\langle T(x)\rangle_{x\in\omega}$ if for every $x \in \operatorname{dom} f$, $f(x) \in T(x)$. Various traceability notions are arrived at by placing size restrictions or effectiveness restrictions on the sequence $\langle T(x)\rangle_{x\in\omega}$. For example:

Definition 1. A set A is *computably traceable* if every total function $f \leqslant_T A$ is traced by a sequence $\langle T(x)\rangle_{x\in\omega}$ with $|T(x)| = x + 1$ and canonical indices for the $T(x)$ can be given computably in x.

The computably traceable sets turn out to be precisely the sets which are low for Schnorr randomness (this is the above mentioned characterization). The characterization of low for Demuth randomness is similar, although slightly more complicated to state.

Towards a combinatorial characterization of K-triviality, jump traceability has been put forth as a candidate.

Definition 2. A *computable order* is a total, computable, non-decreasing, unbounded function.

A set A is *jump traceable at order h* if every partial A-computable function f^A is traced by a sequence $\langle T(x)\rangle_{x\in\omega}$ with $|T(x)| \leqslant h(x)$ and c.e. indices for the $T(x)$ can be given computably in x.

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This is actually many traceability notions, depending on the choice of order h. Clearly slower growing orders result in a more restrictive notion. So we are left with studying the relationship with K-triviality for different orders h. Cholak, Downey and Greenberg [2] showed that every c.e. set which is jump traceable at order $\sqrt{\log x}/9$ is K-trivial. Meanwhile, Hölzl, Kräling and Merkle [4] showed a result which has the following as a corollary:

Proposition 3. *For every K-trivial set A, there is a constant d such that A is jump traceable at order $d \log x$.*

In this paper, we show that this result is, basically, tight. That is, we construct a K-trivial c.e. set which is not jump traceable at any computable order $h \in o(\log x)$. So while the question of a jump traceability characterization for K-triviality remains open, we know that, up to a multiplicative constant, the only possible candidate is $\log x$.

2. Preliminaries

The important property of orders in $o(\log x)$, which this proof depends upon, is the following:

Lemma 4. *If h is an order with $\sum_{x=0}^{\infty} 2^{-d \cdot h(x)} = \infty$ for all $d > 0$, then for any $c, N > 0$, there is an $n > N$ with $h(N + 2^{c \cdot n}) \leqslant n$.*

Note that every order in $o(\log x)$ satisfies the hypothesis; however, these are not the only orders which do. For example, any order in $o(\log x + \log \log x)$ does, as well.

Proof. Fix c and N, and suppose $h(2^{(c+1) \cdot n}) > n$ for all but finitely many n. Then $h(x) > \frac{1}{c+1} \log x$ for all but finitely many x, and thus $\sum_{x=0}^{\infty} 2^{-2(c+1) \cdot h(x)} < \infty$, contrary to hypothesis. So $h(2^{(c+1) \cdot n}) \leqslant n$ for arbitrarily large n. But for sufficiently large n, $n > N$ and $2^{(c+1) \cdot n} > N + 2^{c \cdot n}$. □

Before we move on to the proof of the main result, we discuss the following function:

$$c(z, s) = \Omega_s - \Omega_z.$$

Here Ω is Chaitin's halting probability, and $\langle \Omega_s \rangle_{s \in \omega}$ is some computable increasing sequence of nonnegative rational numbers which converges to Ω. The reader may recognize this as an example of a cost function; cost functions are well studied, particularly in their relation to K-triviality (for a detailed treatment, see Nies's upcoming manuscript [6]). We do not discuss them in generality here, but instead only restate the result we need.

Definition 5. For $\langle A_s \rangle_{s \in \omega}$ an approximation to a Δ_2^0 set A, we define $c(A_s)$ to be $c(z_s, s)$, where z_s is least with $A_s(z_s) \neq A_{s+1}(z_s)$. If $A_s = A_{s+1}$, we take $c(A_s) = 0$.

We say that a set A *obeys c* if it has some computable approximation $\langle A_s \rangle_{s \in \omega}$ with $\sum_s c(A_s) < \infty$.

The important result is the following:

Lemma 6. *(See Nies [6].) For a Δ_2^0 set A, A obeys c if and only if A is K-trivial.*

We will need one other fact:

Observation 7. For a sequence $z_0 < s_0 < z_1 < s_1 < \cdots < z_n < s_n$,

$$\sum_{i=0}^{n} c(z_i, s_i) < 1.$$

This follows by rearranging the terms in the sum:

$$\sum_{i=0}^{n} c(z_i, s_i) = \Omega_{s_n} - \Omega_{z_0} + \sum_{i=0}^{n-1} (\Omega_{s_i} - \Omega_{z_{i+1}}) \leqslant \Omega_{s_n} < \Omega < 1.$$

3. Proof of theorem

Now we are ready to prove the theorem.

Theorem 8. *There is a c.e. K-trivial set A which is not jump traceable at any computable order h with $\sum_{x=0}^{\infty} 2^{-d \cdot h(x)} = \infty$ for all $d > 0$. In particular, it is not jump traceable at any computable order $h \in o(\log x)$.*

Proof. We construct A in stages, ensuring that the approximation we construct obeys c, which will be sufficient to guarantee that A is K-trivial.

Let h^e be an enumeration of all (partial) computable orders, and let $\langle T_k^e(x) \rangle_{k \in \omega}$ be an enumeration of all h^e bounded c.e. traces. We construct a partial A-computable function f^A, and for every $k, e \in \omega$ we meet the following requirements:

P_k^e: If h^e is total with $\sum_{x=0}^{\infty} 2^{-d \cdot h^e(x)} = \infty$ for all $d > 0$, then there is an $x \in \omega$ with $f^A(x)\downarrow$ and $f^A(x) \notin T_k^e(x)$.

We partition the domain of f^A into infinitely many sets B^e, and work to meet all P_k^e-requirements on B^e. However, our choice of partition matters: for reasons that will become apparent later, each B^e must be an arithmetic progression. So we let $B^e = \{n \cdot 2^{e+1} + 2^e \mid n \in \omega\}$.

Basic strategy for P_k^e

The basic strategy for meeting requirement P_k^e is straightforward:

1. Choose an $x \in B^e$.
2. Wait until $h^e(x)\downarrow$.
3. Define $f^A(x)$ to a large value with large use.
4. Wait until $f^A(x) \in T_k^e(x)$.
5. Enumerate an element into A below the use and redefine $f^A(x)$ to a large value with large use.
6. Return to Step 4.

Since $|T_k^e(x)| \leqslant h^e(x)$, the above strategy can only return to Step 4 at most $h^e(x)$ many times, and will eventually wait forever at Step 4 (or instead wait forever at Step 2, in which case $h^e(x)$ is partial). Thus T_k^e will not trace $f^A(x)$, and the requirement will be satisfied.