Information Processing Letters 112 (2012) 406–410

Contents lists available at SciVerse ScienceDirect

Information Processing Letters

www.elsevier.com/locate/ipl

On rainbow-k-connectivity of random graphs ☆

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ARTICLE INFO

Article history:
Received 23 December 2010
Received in revised form 11 January 2012
Accepted 31 January 2012
Available online 1 February 2012
Communicated by F.Y.L. Chin

Keywords:
Rainbow connectivity
Random graph
Graph algorithms
Sharp threshold function
Probabilistic method

ABSTRACT

A path in an edge-colored graph is called a *rainbow path* if the edges on it have distinct colors. For $k \geqslant 1$, the *rainbow-k-connectivity* of a graph G, denoted by $rc_k(G)$, is the minimum number of colors required to color the edges of G in such a way that every two distinct vertices are connected by at least k internally vertex-disjoint rainbow paths. In this paper, we study rainbow-k-connectivity in the setting of random graphs. We show that for every fixed integer $d \geqslant 2$ and every $k \leqslant O(\log n)$, $p = (\log n)^{1/d}/n^{(d-1)/d}$ is a sharp threshold function for the property $rc_k(G(n,p)) \leqslant d$. This substantially generalizes a result in [Y. Caro, A. Lev, Y. Roditty, Z. Tuza, R. Yuster, On rainbow connection, Electron. J. Comb. 15 (2008)], stating that $p = \sqrt{\log n/n}$ is a sharp threshold function for the property $rc_1(G(n,p)) \leqslant 2$. As a by-product, we obtain a polynomial-time algorithm that makes $G(n,p)$ rainbow-k-connected using at most one more than the optimal number of colors with probability $1-o(1)$, for all $k \leqslant O(\log n)$ and $p = n^{-\epsilon(1\pm o(1))}$ for any constant $\epsilon \in [0,1)$.

1. Introduction

All graphs considered in this paper are finite, simple, undirected and contain at least 2 vertices. We follow the notation and terminology of [3]. The following notion of *rainbow-k-connectivity* was proposed by Chartrand et al. [8,9] as a strengthening of the canonical connectivity concept in graphs. Given an edge-colored graph G, a path in G is called a *rainbow path* if its edges have distinct colors. For an integer $k \geqslant 1$, an edge-colored graph is called *rainbow-k-connected* if any two different vertices of G are connected by at least k internally vertex-disjoint rainbow paths. The *rainbow-k-connectivity* of G, denoted by $rc_k(G)$, is the minimum number of colors required to color the edges of G to make it rainbow-k-connected. Note that such coloring does not exist if G is not k-vertex-connected, in which case we simply let $rc_k(G) = \infty$. When $k = 1$ it is alternatively called *rainbow-connectivity* or *rainbow connection number* in literature, and is conventionally written as $rc(G)$ with the subscript k dropped.

Besides its theoretical interest as being a natural combinatorial concept, rainbow connectivity also finds applications in networking and secure message transmitting [6,11,15]. The following motivation is given in [6]: Suppose we want to route messages in a cellular network such that each link on the route between two vertices is assigned with a distinct channel. Then the minimum number of used channels is exactly the rainbow-connectivity of the underlying graph.

Some easy observations regarding rainbow-k-connectivity include that $rc_k(G) = 1$ if and only if $k = 1$ and G is a clique, that $rc(G) \leqslant n-1$ for all connected G, and that $rc(G) = n-1$ if and only if G is a tree, where n is the number of vertices in G. Chartrand et al. [8] determined the rainbow-connectivity of several special classes of graphs, including complete multipartite graphs. In [9] they investigated rainbow-k-connectivity in complete graphs and regular complete bipartite graphs. The extremal graph-theoretic aspect of rainbow-connectivity was studied by Caro et al. [5], who proved that $rc(G) = O_\delta(n \log\delta/\delta)$ with δ being the minimum degree of G. This tradeoff was later improved to

☆ This work was supported in part by the National Basic Research Program of China Grants 2011CBA00300, 2011CBA00301, and the National Natural Science Foundation of China Grants 61033001, 61061130540, 61073174.

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doi:10.1016/j.ipl.2012.01.014

$rc(G) < 20n/\delta$ by Krivelevich and Yuster [13], and was recently shown to be $rc(G) \leqslant 3n/(\delta+1)+3$ by Chandran et al. [7] which is essentially tight. Chakraborty et al. [6] studied the computational complexity perspective of this notion, proving among other results that given a graph G deciding whether $rc(G) = 2$ is NP-complete.

Another important setting that has been extensively explored for studying various graph concepts is the Erdős–Rényi random graph model $G(n, p)$ [10], in which each of the $\binom{n}{2}$ pairs of vertices appears as an edge with probability p independently from other pairs. We say an event $\mathcal{E}$ happens *almost surely* if the probability that it happens approaches 1 as $n \to \infty$, i.e., $\mathbf{Pr}[\mathcal{E}] = 1 - o_n(1)$. We will always assume that n is the variable that tends to infinity, and thus omit the subscript n from the asymptotic notations. For a graph property P, a function $p(n)$ is called a *threshold function* of P if:

- for every $r(n) = \omega(p(n))$, $G(n, r(n))$ almost surely satisfies P; and
- for every $r'(n) = o(p(n))$, $G(n, r'(n))$ almost surely does not satisfy P.

Furthermore, $p(n)$ is called a *sharp threshold function* of P if there exist two positive constants c and C such that:

- for every $r(n) \geqslant C \cdot p(n)$, $G(n, r(n))$ almost surely satisfies P; and
- for every $r'(n) \leqslant c \cdot p(n)$, $G(n, r'(n))$ almost surely does not satisfy P.

Clearly a sharp threshold function of a graph property is also a threshold function of it; yet the converse may not hold, e.g., the property of containing a triangle [2].

It is known that every non-trivial monotone graph property possesses a threshold function [4,12]. Obviously for every k, d, the property $rc_k(G) \leqslant d$ is monotone, and thus has a threshold. Caro et al. [5] proved that $p = \sqrt{\log n/n}$ is a sharp threshold function for the property $rc_1(G(n, p)) \leqslant 2$. In this paper, we significantly extend their result by establishing sharp thresholds for the property $rc_k(G(n, p)) \leqslant d$ for all constants d and logarithmically increasing k. Our main theorem is as follows.

Theorem 1. *Let $d \geqslant 2$ be a fixed integer and $k = k(n) \leqslant O(\log n)$. Then $p = (\log n)^{1/d}/n^{(d-1)/d}$ is a sharp threshold function for the property $rc_k(G(n, p)) \leqslant d$.*

We also investigate rainbow-k-connectivity from the algorithmic point of view. The NP-hardness of determining $rc(G)$ is shown by Chakraborty et al. [6]. We show that the problem (even the search version) becomes easy in random graphs, by designing an algorithm for coloring random graphs to make it rainbow-k-connected with near-optimal number of colors.

Theorem 2. *For any constant $\epsilon \in [0, 1)$, $p = n^{-\epsilon(1\pm o(1))}$ and $k \leqslant O(\log n)$, there is a randomized polynomial-time algorithm that, with probability $1 - o(1)$, makes $G(n, p)$ rainbow-k-connected using at most one more than the optimal number of colors, where the probability is taken over both the randomness of $G(n, p)$ and that of the algorithm.*

Our result is quite strong, since almost all natural edge probability functions p encountered in various scenarios satisfy $p = n^{-\epsilon(1\pm o(1))}$ for some $\epsilon > 0$. Note that $G(n, n^{-\epsilon})$ is almost surely disconnected when $\epsilon > 1$ [10], which makes the problem become trivial. We therefore ignore these cases.

In Section 2 we present the proof of Theorem 1, and in Section 3 we show the correctness of Theorem 2.

2. Threshold of rainbow-*k*-connectivity

This section is devoted to proving Theorem 1. Throughout the paper "ln" denotes the natural logarithm, and "log" denotes the logarithm to the base 2. Hereafter we assume $d \geqslant 2$ is a fixed integer, $c_0 \geqslant 1$ a positive constant, and $k = k(n) \leqslant c_0 \log n$ for all sufficiently large n. To establish a sharp threshold function for a graph property the proof should be two-fold. We first show the easy direction.

Theorem 3. *$rc_k(G(n, (\ln n)^{1/d}/n^{(d-1)/d})) \geqslant d+1$ almost surely holds.*

We need the following fact proved by Bollobás [1].

Lemma 1. *(See restatement of part of Theorem 6 in [1].) Let c be a positive constant and $d \geqslant 2$ a fixed integer. Let $p' = (\ln(n^2/c))^{1/d}/n^{(d-1)/d}$. Then,*

$$\lim_{n\to\infty} \mathbf{Pr}\big[G\big(n, p'\big) \text{ has diameter at most } d\big] = e^{-c/2}.$$

Proof of Theorem 3. Fix an arbitrary $\epsilon > 0$ and choose a constant $c > 0$ so that $e^{-c/2} < \epsilon/2$. Let $p' = (\ln(n^2/c))^{1/d}/n^{(d-1)/d}$ and $p = (\ln n)^{1/d}/n^{(d-1)/d}$. Clearly $p \leqslant p'$ for all $n > c$.

By Lemma 1 and the definition of limits, there exists an $N_1 > 0$ such that for all $n > N_1$, $\mathbf{Pr}[G(n, p')$ has diameter at most $d] < e^{-c/2} + \epsilon/2 < \epsilon$, by our choice of c. Thus, for every $n > \max\{c, N_1\}$,

$$\mathbf{Pr}\big[G(n, p) \text{ has diameter at most } d\big] \leqslant \mathbf{Pr}\big[G\big(n, p'\big) \text{ has diameter at most } d\big] < \epsilon.$$

Due to the arbitrariness of ϵ, this implies that the probability of $G(n, p)$ having diameter at most d is $o(1)$. This completes the proof of Theorem 3, since the rainbow-k-connectivity of a graph is at least as large as its diameter. □

We are left with the other direction stated below. Fix $C = 2^{20} \cdot c_0$.

Theorem 4. *$rc_k(G(n, C(\log n)^{1/d}/n^{(d-1)/d})) \leqslant d$ almost surely holds.*

The key component of our proof of Theorem 4 is the following theorem.