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On the average complexity for the verification of compatible sequences

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A B S T R A C T

The average complexity analysis for a formalism pertaining pairs of compatible sequences is presented. The analysis is done in two levels, so that an accurate estimate is achieved. The way of separating the candidate pairs into suitable classes of ternary sequences is interesting, allowing the use of fundamental tools of symbolic computation, such as holonomic functions and asymptotic analysis to derive an average complexity of $O(n\sqrt{n}\log n)$ for sequences of length n.

1. Introduction

In this paper, we detail an average complexity analysis for a formalism that exhibits the cross-fertilization of combinatorics with theoretical computer science, in the study of sequences with constant (non-)periodic autocorrelation function.

In what follows we present the asymptotic analysis for the average case complexity of deciding if two sequences have *PAF*/*NPAF* (see Definition 1) equal to a constant α. We employ a fine grained analysis initially and then we provide the asymptotics for a pair of sequences of length n. In the course of the analysis we extensively use tools from Computer Algebra and especially tools for dealing with holonomic functions and their asymptotics since such functions occur in the summations needed for the analysis.

In Section 2 we provide the necessary definitions and previous work. In Section 3 we describe the algorithms and the essential part of the theory behind them. In Section 4 we detail the asymptotic analysis for the average case complexity of the algorithms. In Section 5 we conclude by giving some hints on the practical complexity and arguing on the efficiency of the algorithms examined.

2. Preliminaries

This section gives a collection of notions that are used throughout the paper.

Definition 1. For a sequence $A = [a_1, a_2, \dots, a_n]$ of length n the periodic autocorrelation function (*PAF*) and the non-periodic autocorrelation function (*NPAF*), denoted by $P_A(s)$ and $N_A(s)$, are defined as

$$P_A(s) = \sum_{i=1}^{n} a_i a_{i+s}, \quad s = 0, 1, \dots, n-1, \quad \text{and}$$

$$N_A(s) = \sum_{i=1}^{n-s} a_i a_{i+s}, \quad s = 0, 1, \dots, n-1,$$

respectively, where in *PAF* we consider $(i+s)$ modulo n, see also [11,13].

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Definition 2. Two sequences, $A = [a_1, \ldots, a_n]$ and $B = [b_1, \ldots, b_n]$, of length n are said to have *PAF* (respectively *NPAF*) equal to α, if $P_A(s) + P_B(s) = \alpha$ (respectively $N_A(s) + N_B(s) = \alpha$) for $s = 1, \ldots, n-1$.

Following [5] the sequences A and B will be called compatible, if α is a constant. Note that such pairs of sequences are said to have constant (non-)periodic autocorrelation function even though it is the sum of their autocorrelations that is a constant.

We are interested in bundling together the indices of entries with the same sign. This motivates the following definitions:

Definition 3. The positive and negative supports of a sequence $A = [a_1, \ldots, a_n]$, denoted by $POS(A)$ and $NEG(A)$, respectively, are defined as

$$POS(A) = \{i\colon a_i > 0,\ i = 1, \ldots, n\} \quad \text{and}$$

$$NEG(A) = \{j\colon a_j < 0, j = 1, \ldots, n\},$$

while its weight $w(A)$ is defined as $w(A) = |POS(A)| + |NEG(A)|$.

Definition 4. We define the occurrences counting function $[S]_e$ for a multiset S and an element from the domain of elements of S as $[S]_e = |[x \in S\colon\ x = e]|$.

For example, let S be the multiset $S = [1, 1, 2, 2, 2, 4]$. Then $[S]_1 = 2$, $[S]_2 = 3$, $[S]_3 = 0$ and $[S]_4 = 1$.

Lemma 1. *Let A, B be multisets and by $A \uplus B$ denote the union of A and B, preserving all multiplicities. Then $[A \uplus B]_e = [A]_e + [B]_e$.*

For prior usage of multisets in the study of sequences with constant (non-)periodic autocorrelation function we refer to [7,15,20], while for related operations on them see [12].

3. A combinatorial algorithm for compatible sequences

In this section, we present an algorithm that based on their support decides if two sequences are compatible. The use of the support of sequences first appeared in [5] and [6], and recently for such computations in [7] and [15].

3.1. Sets of differences

In order to state the algorithm, we need to define multisets of differences on their support.

Definition 5. We define the following multisets:

- Signed differences in the positive support of A: $D_A^+ = [x - y\colon\ x, y \in POS(A), x > y]$.
- Signed differences in the negative support of A: $D_A^- = [x - y\colon\ x, y \in NEG(A), x > y]$.
- Cross differences between the positive and negative supports of A: $C_A = D_A^{\pm} \uplus D_A^{\mp}$, where $D_A^{\pm} = [x - y\colon\ x \in POS(A), y \in NEG(A), x > y]$ and $D_A^{\mp} = [x - y\colon\ x \in NEG(A), y \in POS(A), x > y]$.

Remark 1. Depending on whether we are interested in *PAF* or *NPAF*:

- in the definition of the sets above, for *PAF* we use $(x - y) \pmod n$ instead of $(x - y)$;
- the notation $AF_A(s)$ denotes either $PAF_A(s)$ or $NPAF_A(s)$,

Notation 1. We denote by $AF_{A,B}(s)$ the sum $AF_A(s) + AF_B(s)$. If the sequences A and B are clear from the context, we denote $AF_{A,B}(s)$ by $AF(s)$.

The motivation behind the multisets in Definition 5 is counting the contribution of each sequence in $AF_{A,B}(s)$.

Lemma 2. *Let A, B be two sequences of length n, weight w and entries from $\{-1, 0, 1\}$. Let D be $D_A^+ \uplus D_A^- \uplus D_B^+ \uplus D_B^-$ and C be $C_A \uplus C_B$. For $s \in \{1, 2, \ldots, n-1\}$, the following are equivalent:*

(i) $AF_{A,B}(s) = \alpha$;
(ii) $[D]_s - [C]_s = \alpha$.

Proof. Fix $s \in \{1, 2, \ldots, n-1\}$. We prove the lemma for *NPAF*. The *PAF* case is similar. We have that $AF_{A,B}(s) = AF_A(s) + AF_B(s)$. Thus we need to compute $AF_C(s)$ for $C = A, B$. We have that $AF_C(s) = |P_C| - |N_C|$, where $P_C = [(c_i, c_{i+s})\colon\ c_i c_{i+s} = 1, i = 1, 2, \ldots, n-s, c_i \in C]$ and $N_C = [(c_i, c_{i+s})\colon\ c_i c_{i+s} = -1, i = 1, 2, \ldots, n-s, c_i \in C]$.

We should consider only elements of the support, since all products involving a zero element contribute zero to the sum. The pairs of elements of the support contributing a "+1" are the ones where both elements have the same sign, i.e., elements of P_C. By the definition of D_C^+ (respectively D_C^-), for every occurrence of s in D_C^+ (respectively D_C^-), there is a pair (c_i, c_{i+s}) for some i such that c_i and c_{i+s} have both positive (respectively negative) sign, thus belonging to P_C. The other direction, i.e., for each pair $(c_i, c_{i+s}) \in P_C$ there is an occurrence of s in D_C^+ or D_C^- follows directly by the definition of D_C^+ and D_C^-. Thus the cardinality of P_C is equal to the number of occurrences of s in D_C^+ and D_C^-.

Similarly, the pairs of elements of the support contributing a "−1" are the ones where elements have opposite signs. By the definition of C_C, for every occurrence of s in C_C, there is a pair (c_i, c_{i+s}) for some i such that c_i and c_{i+s} have opposite signs, thus belonging to N_C. The other direction, i.e., for each pair $(c_i, c_{i+s}) \in N_C$ there is an occurrence of s in C_C follows directly by the definition of C_C. Thus the cardinality of N_C is equal to the number of occurrences of s in C_C.

Summarizing for A and B we have

$$\begin{aligned} AF_{A,B}(s) &= AF_A(s) + AF_B(s) \\ &= |P_A| - |N_A| + |P_B| - |N_B| \end{aligned}$$