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Acyclic chromatic index of planar graphs with triangles

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ABSTRACT

A proper edge coloring of a graph G is called acyclic if there is no 2-colored cycle in G. The acyclic chromatic index of G, denoted by $\chi'_a(G)$, is the least number of colors in an acyclic edge coloring of G. Let G be a planar graph with maximum degree $\Delta(G)$. In this paper, we show that $\chi'_a(G) \leqslant \Delta(G)+4$, if G contains no 4-cycle; $\chi'_a(G) \leqslant \Delta(G)+5$, if G contains no intersecting triangles; and $\chi'_a(G) \leqslant \Delta(G)+6$ if G contains no adjacent triangles.

1. Introduction

In this paper, all graphs considered are finite, simple and undirected. We use $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ to denote the vertex set, the edge set, the minimum degree and the maximum degree of a graph G, respectively. For a vertex $v \in V(G)$, let $N(v)$ denote the set of vertices adjacent to v and $d_G(v) = |N(v)|$ (or $d(v)$ for simple) denote the *degree* of v. A d-, d^+- or d^--*vertex* is a vertex of degree d, at least d or at most d, respectively. The *length* or *degree* of a face f, denoted by $d(f)$, is the length of a boundary walk around f. In particular, a cut-edge is counted twice. A d-, d^+- or d^--*face* or *cycle* is a face or cycle of length d, at least d or at most d, respectively. As usual $[k]$ stands for the set $\{1,\ldots,k\}$.

A *proper edge k-coloring* of a graph G is a mapping ϕ from $E(G)$ to the color set $[k]$ such that no pair of incident edges are colored with the same color. A proper edge coloring of a graph G is called *acyclic* if there is no 2-colored cycle in G. In other words, the union of any two color classes induces a subgraph of G which is a forest. The *acyclic chromatic index* of G, denoted by $\chi'_a(G)$, is the least number of colors in an acyclic edge coloring of G.

Acyclic edge coloring has been widely studied over the past twenty years. The first general linear upper bound on $\chi'_a(G)$ was found by Alon et al. [1] who proved that $\chi'_a(G) \leqslant 64\Delta(G)$. This bound was improved to $16\Delta(G)$ by Molloy and Reed [15].

In 2001, Alon, Sudakov and Zaks [2] stated the Acyclic Edge Coloring Conjecture, which says that $\chi'_a(G) \leqslant \Delta(G)+2$ for every graph G. This conjecture was verified for some special classes of graphs, including non-regular graphs with maximum degree at most four [5], subcubic graphs [4,18], outerplanar graphs [13,17], series-parallel graphs [12], grid like graphs [16], graphs with large girth [2], and so on.

Fiedorowicz et al. [10] gave an upper bound of $2\Delta(G)+29$ for planar graphs and of $\Delta(G)+6$ for triangle-free planar graphs. Independently, Hou et al. [12] proved that $\chi'_a(G) \leqslant \max\{2\Delta(G)-2, \Delta(G)+22\}$ for planar graphs and $\chi'_a(G) \leqslant \Delta(G)+2$ for planar graphs with girth at least 5, where the girth of a graph is the length of a shortest cycle in it. For planar graphs with girth at least 5 and

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with maximum degree at least 10, $\chi'_a(G) = \Delta(G)$ [11,14]. For triangle-free planar graphs, Basavaraju and Chandran [6] proved that $\chi'_a(G) \leqslant \Delta(G) + 3$.

Borowiecki and Fiedorowicz [7] showed that $\chi'_a(G) \leqslant \Delta(G) + 15$ for planar graphs without cycles of length 4. The first result of this paper improves the bound to $\Delta(G) + 4$.

Theorem 1. *Let G be a planar graph without 4-cycles. Then $\chi'_a(G) \leqslant \Delta(G) + 4$.*

The following two results are for planar graphs without close triangles. Two triangles are *intersecting* if they have at least one vertex in common. Two triangles are *adjacent* if they have at least one edge in common.

Theorem 2. *Let G be a planar graph without intersecting triangles. Then $\chi'_a(G) \leqslant \Delta(G) + 5$.*

Theorem 3. *Let G be a planar graph without adjacent triangles. Then $\chi'_a(G) \leqslant \Delta(G) + 6$.*

2. Critical graphs

Let $\phi : E(G) \to [k]$ be an edge k-coloring of G. For a vertex $v \in V(G)$ and an edge e incident with v, we say that the color $\phi(e)$ *appears* on v. Let $C_\phi(v) = \{\phi(uv) \mid u \in N(v)\}$ and $S_\phi(v) = [k] - C_\phi(v)$. An (α, β)-path is a path consisting of edges that are colored α and β alternately. A graph G is called an *acyclically edge k-critical graph* if $\chi'_a(G) > k$ and every proper subgraph of G is acyclically edge k-colorable. In particular, such a graph has maximum degree at most k.

Lemma 1. *If G is an acyclically edge k-critical graph, then G is 2-connected.*

Proof. By contradiction, let v be a cut vertex of G and let $C_1, \ldots, C_t$ $(t \geqslant 2)$ be the connected components of $G - v$. By the choice of G, for each $1 \leqslant i \leqslant t$, there is an acyclic edge coloring ϕ_i of $G_i = C_i \cup \{v\}$ with k colors. We can permute the colors in each ϕ_i (if needed) such that the colors appearing on the edges incident with v are all different. Now the union of these colorings will be an acyclic edge coloring of G. □

Lemma 2. *Let G be an acyclically edge k-critical graph and v be a d-vertex of G with neighbors $x_1, \ldots, x_d$, then $\sum_{i=1}^{d} d(x_i) \geqslant k + d$.*

Proof. Suppose to the contrary that $\sum_{i=1}^{d} d(x_i) \leqslant k + d - 1$. Then the graph $G' = G - v$ admits an acyclic edge k-coloring ϕ by the choice of G. Assume $d(x_1) \geqslant \cdots \geqslant d(x_d) \geqslant 2$ by Lemma 1 and $A_i = \bigcup_{j=i}^{d} C_\phi(x_j)$ for $i \in \{1, \ldots, d\}$. Then

$$|A_i| \leqslant \sum_{j=i}^{d} \bigl(d(x_j) - 1\bigr) = \sum_{j=1}^{d} \bigl(d(x_j) - 1\bigr) - \sum_{j=1}^{i-1} \bigl(d(x_j) - 1\bigr) \leqslant \sum_{j=1}^{d} \bigl(d(x_j) - 1\bigr) - (i - 1) \leqslant k - i$$

for $i \in \{2, \ldots, d\}$. We can choose a color $\alpha_1 \notin A_1$ to color vx_1 and a color $\alpha_i \notin A_i$ and different from $\alpha_1, \ldots, \alpha_{i-1}$ to color vx_i for $i \in \{2, \ldots, d\}$. This produces a proper edge coloring of G. Now we show that there is no 2-colored cycle. Cycles that do not contain v have edges of at least three colors as the edge coloring of G' was acyclic. Now a cycle containing v must contain the edges vx_i, vx_j, say $i < j$ and an edge $x_j w$ with $w \in N(x_j)$. By construction, the colors on $vx_i, vx_j, x_j w$ are distinct. Thus the resulting coloring is an acyclic edge k-coloring of G, which is a contradiction. □

Corollary 4. *Let G be an acyclically edge k-critical graph. If $k \geqslant 2\Delta(G) - 1$, then G has no 2-vertices. If $k \leqslant 2\Delta(G) - 2$, then the neighbors of every 2-vertex of G have degree at least $k - \Delta(G) + 2$.*

Lemma 3. *Let G be an acyclically edge k-critical graph with $k \leqslant 2\Delta(G) - 2$ and v be a vertex of G adjacent to a 2-vertex w. Then v is adjacent to at least $k - \Delta(G) + 1$ vertices of degree at least $k - \Delta(G) + 2$.*

Proof. Suppose to the contrary that v is adjacent to at most $k - \Delta(G)$ vertices of degrees at least $k - \Delta(G) + 2$. By the choice of G, $G - w$ admits an acyclic edge k-coloring ϕ. Let u be the other neighbor of w. Then $|S_\phi(u)| = k - (d(u) - 1) \geqslant k - \Delta(G) + 1$. Note that every color in $S_\phi(u)$ must appear on the vertex v, since otherwise, we can color uw with a color $\alpha \in S_\phi(u) - C_\phi(v)$ and choose a color β different from α and not appearing on v to color wv. Since v is adjacent to at most $k - \Delta(G)$ vertices of degrees at least $k - \Delta(G) + 2$, there exists a $(k - \Delta(G) + 1)^-$-vertex in $N(v)$, say x, such that $\phi(vx) \in S_\phi(u)$. Then we color wu with $\phi(vx)$ and color vw with a color neither in $C_\phi(v)$ nor in $C_\phi(x)$, this is possible since $|C_\phi(v) \cup C_\phi(x)| \leqslant d(v) - 2 + d(x) \leqslant k - 1$. □

Lemma 4. *Let G be an acyclically edge k-critical graph with $k \geqslant \Delta(G) + 2$ and v be a 3-vertex of G. Then the neighbors of v have degree at least $k - \Delta(G) + 2$.*

Proof. Suppose to the contrary that v is adjacent to a $(k - \Delta(G) + 1)^-$-vertex x. Assume that $N(v) = \{x, y, z\}$, $N(x) = \{v, x_1, \ldots, x_t\}$, where $t \leqslant k - \Delta(G)$. The graph $G' = G - vx$ admits an acyclic edge k-coloring ϕ. We may assume that $\phi(xx_i) = i$ for $i \in \{1, \ldots, t\}$. If $i \notin C_\phi(v)$ for all $i \in \{1, \ldots, k - \Delta(G)\}$, then we can color vx with a color appearing neither on v nor on x. Otherwise, we have $\{1, \ldots, t\} \cap C_\phi(v) \neq \emptyset$. Without loss of generality, let $\phi(vy) = 1$.

Case 1. $\phi(vz) \notin \{1, \ldots, t\}$.

Let $\phi(vz) = t + 1$. Then, for $t + 2 \leqslant i \leqslant k$, there is a $(1, i)$-path from v to x through y and x_1, since otherwise, we can color vx with i. This implies that $t = k - \Delta(G)$, and $C_\phi(y) = \{1, k - \Delta(G) + 2, \ldots, k\}$. If there exists a color in $\{k - \Delta(G) + 2, \ldots, k\}$, say α, such that $\alpha \in S_\phi(z)$, then we recolor vz with α and color vx with $t + 1$. Since there is a $(1, \alpha)$-path from v to x through y, there is no $(1, \alpha)$-path from v to z through y. This implies that the resulting coloring is an acyclic edge coloring of G, a contradiction.