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Online uniformity of integer points on a line ☆

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ABSTRACT

This Letter presents algorithms for computing a uniform sequence of n integer points in a given interval $[0, m]$ where m and n are integers such that $m > n > 0$. The uniformity of a point set is measured by the ratio of the minimum gap over the maximum gap. We prove that we can insert n integral points one by one into the interval $[0, m]$ while keeping the uniformity of the point set at least $1/2$. If we require uniformity strictly greater than $1/2$, such a sequence does not always exist, but we can prove a tight upper bound on the length of the sequence for given values of n and m.

1. Introduction

A number of applications need uniformly distributed points over a specific domain. It is commonly known that randomly generated points are not always good enough. In a mesh generation, for example, we have to distribute points uniformly over a region of interest to form good meshes. But, first of all, how can we measure the uniformity of points? In the theory of Discrepancy [2,3] the uniformity of points is measured by how the number of points in a small region such as an axis-parallel rectangle changes while moving around the domain, more formally by the difference (or discrepancy) between the largest and smallest numbers of points in the moving region. For normalization we usually divide the difference by the area of the moving region. Then, the discrepancy is given as the supremum of the ratios for all possible scales of the region. One of difficulties here is hardness of such evaluation since we have to prepare all possible scales and all possible locations.

We consider a special case of such a problem, that is, how to insert n integer points in a given interval $[0, m]$ so that points are uniformly distributed, or in other words, the ratio of the minimum gap over the maximum gap is not so low. We present a simple algorithm for achieving the ratio $1/2$ for all integers m and n with $m > n > 0$. It is not trivial at all to achieve the ratio strictly greater than $1/2$. In addition, if we require uniformity strictly greater than $1/2$, such a sequence does not always exist, but we can prove a tight upper bound on the length of the sequence for given values of n and m.

The problem considered in this letter may open a new direction of discrepancy theory. The first extension from the current discrepancy theory is from uniformity measure for a static set of points to one for a sequence of points. The second extension is from continuous coordinates to discrete ones. This extension is important since now we have a discrete combinatorial optimization problem, which may lead to good approximation algorithms.

2. Problem

Let $m > n > 0$ be arbitrary integers. An (n, m)-*sequence* is a sequence of integers (or points of integral coordinates) $\sigma = (0, m, p_1, \ldots, p_n)$ in the closed interval $[0, m]$. The uniformity of the sequence is measured by the ratio of the minimum gap over the maximum gap where a gap is difference between two consecutive integer points when they are arranged on a line in a sorted order. It may be natural and reasonable to measure the uniformity of a point set $\{0, m, p_1, \ldots, p_n\}$ by the ratio of the

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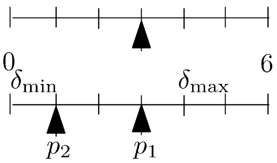

Fig. 1. Behavior of Voronoi insertion on integer points.

minimum gap $\delta_{\min}(0, m, p_1, \ldots, p_n)$ over the maximum gap $\delta_{\max}(0, m, p_1, \ldots, p_n)$, that is, the (*static*) *uniformity* $\mu_s(0, m, p_1, \ldots, p_n)$ of the set is defined by

$$\mu_s(0, m, p_1, \ldots, p_n) = \frac{\delta_{\min}(0, m, p_1, \ldots, p_n)}{\delta_{\max}(0, m, p_1, \ldots, p_n)}. \tag{1}$$

In this Letter we are interested in uniformity achieved by a sequence of points. That is, points are inserted one by one. Every time when a point is inserted, we measure the uniformity of the point set. The worst uniformity we obtain before inserting all the points according to a given point sequence is defined as the *online uniformity* of the point sequence. Formally, we define the *online uniformity* $\mu(0, m, p_1, \ldots, p_n)$ for a point sequence $(0, m, p_1, \ldots, p_n)$ of length n (neglecting the first two points 0 and m) in the interval $[0, m]$ by

$$\mu(0, m, p_1, \ldots, p_n) = \min_{k=1,\ldots,n} \bigl\{\mu_s(0, m, p_1, \ldots, p_k)\bigr\}. \tag{2}$$

We call an (n, m)-sequence *uniform* if its online uniformity is strictly greater than $1/2$.

3. Greedy algorithm

A natural and naive idea to design a uniform sequence of points in a given interval $[0, m]$ is to repeat inserting a point to break the longest interval (of the maximum gap). It is rather straightforward to generalize this idea to higher dimensions. In higher dimensions we construct a Voronoi diagram for a current set of points and choose one of Voronoi vertices that is farthest from the closest point as the next point to insert. Thus, we call the algorithm *Voronoi Insertion*.

The performance of this greedy algorithm is not so bad. In fact, it achieves the uniformity $1/2$ in one dimension [4,1]. However, it is not the case when points are limited to integer points. As a simple example consider a sequence of length 2 for an interval $[0, 6]$ (see Fig. 1). The greedy algorithm chooses 3 as the first point, which is the midpoint of the interval. Then, we have two subintervals of length 3. Since we have to choose only integral points, one of the subintervals is divided into two subintervals of lengths 1 and 2. So, after choosing the two points the minimum gap is 1 while the maximum gap remains 3. So, the uniformity is $1/3 < 1/2$.

As another example consider a case of $(m, n) = (10, 4)$. For simplicity of arguments we just maintain interval lengths instead of intervals. Initially we have $\{10\}$. By the first point we must have $\{5, 5\}$ or $\{4, 6\}$ since otherwise the uniformity would be worse (smaller) than $1/2$. The even partition $\{5, 5\}$ does not lead to uniformity $1/2$ because in the next division we have $\{2, 3, 5\}$, whose uniformity is $2/5 < 1/2$. So, $\{4, 6\}$ is the only choice and then we

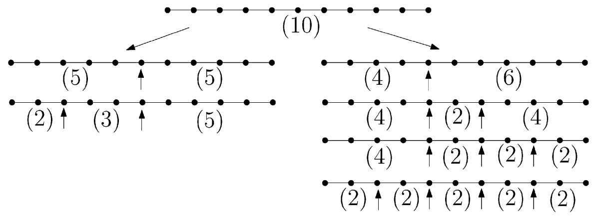

Fig. 2. Partition of an interval of length 10 in two different ways. If we divide 10 into 5, 5 as in the left figure, it is impossible to keep the uniformity $\geqslant 1/2$. The division $10 \to (4, 6)$ leads to a sequence with uniformity $\geqslant 1/2$.

obtain the set $\{4, 2, 4\}$ by dividing the interval of length 6. Now, we can divide 4 into $\{2, 2\}$. Thus, the resulting set of interval lengths is $\{4, 2, 2, 2\}$ with uniformity $2/4 = 1/2$. On the other hand, if we divide 6 into $\{3, 3\}$, we have $\{4, 3, 3\}$. Now there is only one way of dividing the largest gap 4: $4 \to \{2, 2\}$. Then, we have $\{2, 2, 3, 3\}$. We have to divide 3, but there is only one way: $3 \to \{1, 2\}$. Thus, the resulting set is $\{2, 2, 1, 2, 3\}$ with uniformity $1/3 < 1/2$. See Fig. 2 for illustration. This example implies that choosing the midpoint of the longest interval may not be so good even if there is a unique midpoint (note that there are two midpoints in an interval of odd length).

Now, a natural question is whether there is an algorithm for finding a sequence of points with uniformity at least $1/2$ for any pair of integers m and n with $m > n$. The following lemma answers the question in an affirmative way.

Lemma 1. *There is an algorithm for finding an (n, m)-sequence of n points in an interval $[0, m]$ with uniformity at least $1/2$ for any pair of integers m and n if $m > n > 0$.*

Proof. We prove the lemma in a constructive manner. The algorithm iteratively partitions the longest interval (maximum gap). An important thing is to divide an interval of length m into ones of lengths 2^k and the rest $r = m - 2^k$ where k is an integer satisfying $3 \times 2^{k-1} \leqslant m < 3 \times 2^k$. If m happens to be a power of 2, say 2^ℓ, then 2^ℓ is partitioned into $\{2^{\ell-1}, 2^{\ell-1}\}$ since in this case we have $k = \ell - 1$. In fact, we have

$$3 \times 2^{\ell-2} \leqslant 4 \times 2^{\ell-2} = 2^\ell = 2 \times 2^{\ell-1} < 3 \times 2^{\ell-1}.$$

Thus, an interval of length 2^k is exactly halved in a way: $2^k \to 2^{k-1} \to \cdots \to 2 \to 1$.

On the other hand, if we have any other integer, then it is partitioned into a power of 2 and the rest in the manner described above. Because of the definition of the partition, the uniformity is at least $1/2$. In fact, if $r = m - 2^k$ is greater than 2^k, then the uniformity is given by

$$2^k/(m - 2^k) \geqslant 2^k/(3 \times 2^k - 2^k) = 1/2,$$

and if r is at most 2^k then it is given by

$$(m - 2^k)/2^k \geqslant (3 \times 2^{k-1} - 2^k)/2^k = 1/2.$$

Thus, dividing the interval of length r is *safe* in the sense that it keeps the uniformity $\geqslant 1/2$. Dividing the interval of length 2^k is also safe since it is divided evenly. □