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TETRAVEX is NP-complete

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TETRAVEX is a widely played one person computer game in which you are given n^2 unit tiles, each edge of which is labelled with a number. The objective is to place each tile within a n by n square such that all neighbouring edges are labelled with an identical number. Unfortunately, playing TETRAVEX is computationally hard. More precisely, we prove that deciding if there is a tiling of the TETRAVEX board given n^2 unit tiles is NP-complete. Deciding where to place the tiles is therefore NP-hard. This may help to explain why TETRAVEX is a good puzzle. This result compliments a number of similar results for one person games involving tiling. For example, NP-completeness results have been show for: the offline version of Tetris [1], KPlumber (which involves rotating tiles containing drawings of pipes to make a connected network) [2], and shortest sliding puzzle problems [3]. It raises a number of open questions. For example, is the infinite version Turing-complete? How do we generate TETRAVEX problems which are truly puzzling as random NP-complete problems are often surprising easy to solve? Can we observe phase transition behaviour? What about the complexity of the problem when it is guaranteed to have an unique solution? How do we generate puzzles with unique solutions?

Theorem 1. TETRAVEX *is* NP-*complete.*

Proof. Clearly it is in NP. Given a solution, we can check it in polynomial time. To show completeness, we use a reduction from 1in3-SAT on purely positive clauses. We will map a problem in n variables and m clauses onto a rectangular TETRAVEX problem of size $O(n)$ by $O(m)$. We can always convert a rectangular TETRAVEX problem into an essentially equivalent but larger square problem by adding suitable tiles.

There are five types of component used in the construction: a horizontal assignment component along the top edge, vertical clause components, vertical and horizontal wiring tiles and junction components to connect vertical to horizontal wires. We assume the variables are labelled from 1 to n. The ith part of the assignment component consists of four tiles that are in one of two configurations:

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$$i = \mathit{true}\colon\ \boxed{\begin{array}{ccc} & \mathit{top} & \\ i-1 & & i \\ & 0 & \end{array}}\ \boxed{\begin{array}{ccc} & \mathit{top} & \\ i & & i \\ & i & \end{array}}\ \boxed{\begin{array}{ccc} & \mathit{top} & \\ i & & i \\ & -i & \end{array}}\ \boxed{\begin{array}{ccc} & \mathit{top} & \\ i & & i+1 \\ & 0 & \end{array}}$$

$$i = \mathit{false}\colon\ \boxed{\begin{array}{ccc} & \mathit{top} & \\ i-1 & & i \\ & 0 & \end{array}}\ \boxed{\begin{array}{ccc} & \mathit{top} & \\ i & & i \\ & -i & \end{array}}\ \boxed{\begin{array}{ccc} & \mathit{top} & \\ i & & i \\ & i & \end{array}}\ \boxed{\begin{array}{ccc} & \mathit{top} & \\ i & & i+1 \\ & 0 & \end{array}}$$

The value *top* is used to ensure that these tiles can only be placed along the top of puzzle. No tile in the puzzle has *top* at its bottom label. Actually, this is not essential and we can label the top of these tiles 0. However, it makes the proof easier if we force the assignment component to be on the top edge of the puzzle. The $i-1$ value in the leftmost tile, and the $i+1$ value in the rightmost tile are used to ensure that the assignment components are laid out in order from left to right along the top row of the puzzle. The value 0 is used for internal tiles which are not components. We also start and end the top row with the tiles:

$$\boxed{\begin{array}{ccc} & \mathit{top} & \\ \mathit{left} & & 0 \\ & 0 & \end{array}}\ \ldots\ \boxed{\begin{array}{ccc} & \mathit{top} & \\ n+1 & & \mathit{right} \\ & 0 & \end{array}}$$

The value *left* and *right* are used to ensure that a tile appears on the left or right edge of the puzzle. No tile has *left* as its right label. Similarly, no tile has *right* as its left label. This is not essential and we could label them zero, but it again makes the proof easier.

This "signal" (i, $-i$ which is interpreted as i is *true*, or $-i$, i which is interpreted as i is *false*) is then transmitted to the vertical clause components via "wires". There are vertical wires, horizontal wires and junctions. A vertical wire is of the form:

$$\boxed{\begin{array}{ccc} & i & \\ X & & X \\ & i & \end{array}}\ \text{and}\ \boxed{\begin{array}{ccc} & -i & \\ X & & X \\ & -i & \end{array}}$$

Where X is either 0 (if the wire is passing a blank part of the puzzle), or the value of a horizontal wire being crossed. It is important to note that this value for X is not equal to i. When we cross a wire carrying the signal from the ith variable, we use a junction component. These vertical wiring tiles can appear in either order depending on the polarity of the signal being transmitted. A horizontal wire is of the form:

$$\boxed{\begin{array}{ccc} & X & \\ i & & i \\ & X & \end{array}}\ \text{and}\ \boxed{\begin{array}{ccc} & X & \\ -i & & -i \\ & X & \end{array}}$$

Where X is again either 0 (if the wire is passing a blank part of the puzzle), or the value of a vertical wire being crossed. We again note that this value for X is not equal to i. These horizontal wiring tiles can appear in either order depending on the polarity of the signal being transmitted.

A junction connects a vertical pair of wires with a horizontal pair of wires. Each junction is labelled with an unique number j where $j > n + m$. The junction consists of one of two possible arrangements of four tiles:

$$\begin{array}{c} \boxed{\begin{array}{ccc} & i & \\ i & & -i \\ & -j & \end{array}}\ \boxed{\begin{array}{ccc} & -i & \\ -i & & i \\ & j & \end{array}} \\ \boxed{\begin{array}{ccc} & -j & \\ -i & & i \\ & i & \end{array}}\ \boxed{\begin{array}{ccc} & j & \\ i & & -i \\ & -i & \end{array}} \end{array}\ \text{or}\ \begin{array}{c} \boxed{\begin{array}{ccc} & -i & \\ -i & & i \\ & j & \end{array}}\ \boxed{\begin{array}{ccc} & i & \\ i & & -i \\ & -j & \end{array}} \\ \boxed{\begin{array}{ccc} & j & \\ i & & -i \\ & -i & \end{array}}\ \boxed{\begin{array}{ccc} & -j & \\ -i & & i \\ & i & \end{array}} \end{array}$$