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A 7/6-approximation algorithm for the minimum 2-edge connected subgraph problem in bipartite cubic graphs

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ABSTRACT

The minimum 2-edge connected spanning subgraph problem in 3-edge connected cubic bipartite graphs is addressed. For the nonbipartite case, the previous best approximation ratio has been 6/5. We exhibit the advantage of bipartiteness to attain an improved ratio 7/6.

1. Introduction

We address the *minimum* 2-*edge connected spanning subgraph problem* (*M2ECSSP*), the objective of which is to find a 2-edge connected spanning subgraph with minimum number of edges in a given 2-edge connected graph. This problem is widely studied in network design, and closely related to the traveling salesman problem (TSP) as well: if a Hamilton cycle exists, then it is an optimal solution. Hence this problem is NP-hard even in 3-connected bipartite cubic graphs [1].

While the M2ECSSP is MAX SNP-hard even in cubic graphs [6], Khuller and Vishkin [10] gave a 3/2-approximation algorithm for this problem in general graphs, followed by a 17/12-approximation algorithm due to Cheriyan, Sebő and Szigeti [4]. The current best ratio is 4/3 due to Sebő and Vygen [11].

Improvements in the approximation ratio in several graph classes are made. For the M2ECSSP in 3-edge connected cubic graphs, Huh [7] gave a 5/4-approximation algorithm, and Boyd, Iwata, and Takazawa [3] gave a further improvement of 6/5-approximation. In subcubic graphs, a recent result of Boyd, Fu, and Sun [2] attains 5/4-approximation.

In this paper we focus on approximation of the M2ECSSP in bipartite cubic graphs. One motivation to deal with this graph class comes from recent intense work on the TSP. Unlike other combinatorial optimization problems such as the matching, covering, and coloring problems, not many results benefitting from bipartiteness have been known for the TSP. Recently, however, several improvements in the graph-TSP in bipartite cubic graphs are presented [5,9,12]. Thus it would be of interest whether bipartiteness is also of advantage in the M2ECSSP.

Indeed, we exhibit advantages of bipartiteness to improve the approximation ratio in 3-edge connected cubic graphs. Namely, our contribution is a 7/6-approximation algorithm for the M2ECSSP in 3-edge connected cubic bipartite graphs, which improves upon the current best ratio 6/5. We employ the ideas in [3], and prove that bipartiteness helps both in improving the approximation ratio and in proving that the algorithm does not get stuck.

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2. Preliminaries

Let $G = (V, E)$ be a simple undirected graph and let $n = |V|$. For a subgraph H of G, the vertex and edge sets of H are denoted by $V(H)$ and $E(H)$, respectively. Let $\delta(H) \subseteq E$ denote the set of edges having exactly one endpoint in $V(H)$. For $X \subseteq V$, let $G[X] = (X, E[X])$ denote the subgraph induced by X. The complement of X is denoted by $\bar{X}$, i.e., $\bar{X} = V \setminus X$.

If every vertex has exactly three incident edges, then G is called *cubic*. A subset F of E is a 2-*factor* if every vertex has exactly two incident edges in F. A *cycle* is defined as a connected subgraph in which each vertex has exactly two incident edges. For a cycle C, the length of C is defined by the number of its edges and denoted by $|C|$. A *path* is a connected subgraph in which every vertex has exactly two incident edges except for two vertices with one incident edge.

An *edge cut* of a connected graph is a minimal subset of edges whose removal makes the graph disconnected. An edge cut of size k is called a k-*edge cut*. If the minimum size of an edge cut in a graph is k, the graph is called k-*edge connected*. In the present paper, we deal with $\{3, 4\}$-*covering* 2-*factors*, defined as 2-factors intersecting all 3- and 4-edge cuts. If G is 2-edge connected and cubic, a $\{3, 4\}$-covering 2-factor always exists [8], and is found in $O(n^3)$ time [3].

3. A 7/6-approximation algorithm

In this section, we describe an algorithm for finding a minimum 2-edge connected spanning subgraph of at most $7n/6 - 1$ edges in 3-edge connected cubic bipartite graphs. For the nonbipartite case, i.e., for 3-edge connected cubic graphs, Boyd, Iwata and Takazawa [3] designed 6/5-approximation algorithms. While our algorithm is mostly the same as an algorithm in [3], a certain difference appears in the definition of “small” and “large” cycles, which directly leads to the improvement of the approximation ratio.

3.1. A rough sketch

Let $G = (V, E)$ be a 3-edge connected cubic bipartite graph. Then G has a $\{3, 4\}$-covering 2-factor F, which is found in $O(n^3)$ time [3,8]. Denote the family of cycles in (V, F) by $\mathcal{C}_F$ and let $C \in \mathcal{C}_F$. We assume that $V(C) \subsetneq V$, since otherwise (V, F) is a Hamilton cycle and we done. Clearly $\delta(C)$ is an edge cut, and since G is 3-edge connected and F is $\{3, 4\}$-covering, we have that $|\delta(C)| \geq 5$. Thus, $|C| \geq 6$ and $|\delta(C)| \geq 6$ follow since G is bipartite and cubic.

Now it is not difficult to attain 4/3-approximation. Contract each cycle in $\mathcal{C}_F$ and denote the resulting graph by G'. We remark that G' is 2-edge connected and has $|\mathcal{C}_F|$ vertices. In G', find a 2-edge connected spanning subgraph H' with at most $2|\mathcal{C}_F| - 2$ edges. This can be done, for example, by finding an ear decomposition and discarding ears consisting of a single edge. Finally, the union of F and the edge set of H' provides a 2-edge connected subgraph of G, which consists of $n + 2|\mathcal{C}_F| - 2 \leq 4n/3 - 2$ edges.

A key idea to improving the approximation ratio is that, for a small cycle C, we add a Hamilton path in $G[V(C)]$ instead of C itself, which saves one edge per one small cycle. The following lemma plays a key role.

Lemma 1 *([3]). Let $G = (V, E)$ be a 2-edge-connected graph and C be a cycle in G with at most two chords. Let $V^* \subseteq V(C)$ be the set of vertices not incident to the chords. For an arbitrary vertex $v^* \in V^*$, there is a Hamilton path in $G[V(C)]$ starting at v^* and ending at some vertex $u^* \in V^*$.*

Since G is cubic, if C has k chords, then $|\delta(C)| = |C| - 2k$. Since $|\delta(C)| \geq 6$, if $|C| \leq 10$, then C has at most two chords and hence Lemma 1 is applied to C. We call a cycle C *small* if $|C| \leq 10$, and *large* if $|C| \geq 12$.

A nontrivial difficulty in this idea is that picking a Hamilton path prescribes the next cycle to visit, and this cycle might be already contained in the current ear. A detailed argument to resolve this difficulty is described in Section 3.2.

We further remark here that the definition of small and large cycles is the difference from [3]. In [3], a cycle C is small if $|C| \leq 9$ and large if $|C| \geq 10$. The reason for this difference is also described in Section 3.2.

3.2. Lollipops and tadpoles

This subsection is intended to an intuitive understanding of the concepts of *lollipops* and *tadpoles* in [3]. A precise definition appears in the algorithm description in Section 3.3.

Recall that G' is a graph obtained by contracting the cycles in $\mathcal{C}_F$. Denote the vertex in G' resulting from a cycle $C \in \mathcal{C}_F$ by v_C. We remark that all procedures are executed in G, but keeping G' in mind shall provide an adequate understanding.

Our purpose is to combine the edges in the large cycles, Hamiltonian paths in the small cycles shown in Lemma 1, and an ear decomposition in G'. However, if we arrive at v_C such that C is a small cycle in finding an ear decomposition in G', the next edge to traverse is determined by Lemma 1, and then the next vertex $v_{C'}$ might be contained in the current ear.

In such a case, we continue to construct the current ear. The construction depends on whether C' is a large cycle or a small cycle.

If C' is a large cycle, construct a lollipop L, which is a subgraph of G consisting of C' and the subgraph traversed so far by the algorithm after it first encountered C'. We then update G' by contracting the vertices v_C such that C is contained in L to create a new vertex v_L, and restart constructing an ear from v_L. See Fig. 1 for an illustration. In Fig. 1, H is a 2-edge connected subgraph consisting of previously constructed ears. The current ear under construction consists of $v_{C_1}, \ldots, v_{C_5}$, where C_1 and C_5 are small cycles of length six, and C_2, C_3 and C_4 are large cycles (some vertices in large cycles are omitted). We have now reached v_{C_2} again, by traversing a Hamilton path in $G[V(C_5)]$, and then we construct a lollipop consisting of thick edges within and connecting C_2, C_3, C_4 and C_5.