



# Topological orderings of weighted directed acyclic graphs

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## ABSTRACT

We call a topological ordering of a weighted directed acyclic graph *non-negative* if the sum of weights on the vertices in any prefix of the ordering is non-negative. We investigate two processes for constructing non-negative topological orderings of weighted directed acyclic graphs. The first process is called a *mark sequence* and the second is a generalization called a *mark–unmark sequence*. We answer a question of Erickson by showing that every non-negative topological ordering that can be realized by a mark–unmark sequence can also be realized by a mark sequence. We also investigate the question of whether a given weighted directed acyclic graph has a non-negative topological ordering. We show that even in the simple case when every vertex is a source or a sink the question is NP-complete.

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## 1. Introduction

A *directed acyclic graph* (or DAG) is a directed graph with no directed cycles. A subset  $M$  of vertices of  $G$  is *outdirected* if every edge between  $M$  and  $V(G) \setminus M$  is directed towards  $V(G) \setminus M$  (i.e., edges directed towards  $M$  are contained in  $M$ ). A *prefix of length  $k$*  of a sequence  $s$  is the subsequence of the first  $k$  terms of  $s$ . A *topological ordering* of a DAG  $G$  is an ordering of the vertices of  $G$  such that

every prefix of the ordering is outdirected. The following two processes yield topological orderings of a given DAG  $G$  with  $n$  vertices.

A *mark sequence* of  $G$  is a sequence  $M_1, M_2, \dots, M_n$  of subsets of  $V(G)$  formed in the following way: first choose an arbitrary source  $v$  and put  $M_1 = \{v\}$ , i.e., *mark  $v$*  in step 1. For  $i = 2, 3, 4, \dots, n$ , choose a vertex  $u \notin M_{i-1}$  such that  $\{u\} \cup M_{i-1}$  is outdirected and put  $M_i = \{u\} \cup M_{i-1}$ , i.e., *mark  $u$*  in step  $i$ .

A *mark–unmark sequence* of  $G$  is a sequence of subsets of  $V(G)$  formed in the following way: first choose an arbitrary source  $v$  and put  $M_1 = \{v\}$ , i.e., *mark  $v$*  in step 1. For  $i = 2, 3, 4, \dots, n$  either (i) choose a vertex  $u \notin M_{i-1}$  such that  $\{u\} \cup M_{i-1}$  is outdirected and put  $M_i = \{u\} \cup M_{i-1}$ , i.e., *mark  $u$*  in step  $i$  or (ii) choose a vertex  $u \in M_{i-1}$  such that  $M_{i-1} \setminus \{u\}$  is outdirected and put  $M_i = M_{i-1} \setminus \{u\}$ , i.e., *unmark  $u$*  in step  $i$ . This process stops when  $M_i = V(G)$ .

Clearly, mark–unmark sequences are a generalization of mark sequences. Because we only mark a vertex if the new set  $M_i$  is outdirected, we get a topological ordering by arranging the vertices of  $G$  by the last step in which they were marked in the mark–unmark sequence. In particular,

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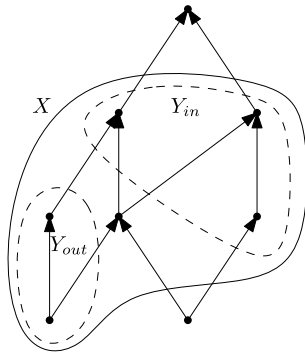
E-mail address: [gerbner.daniel@renyi.mta.hu](mailto:gerbner.daniel@renyi.mta.hu) (D. Gerbner).

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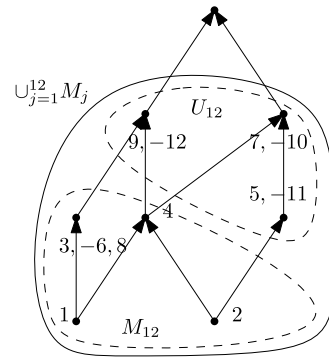
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(a)  $Y_{in}$  is an  $X$ -indirected set,  $Y_{out}$  is an  $X$ -outdirected set.



(b) 12 steps of a mark-unmark sequence, number  $j$  (resp.  $-j$ ) denotes that the vertex was added (resp. removed) in step  $j$ .

Fig. 1. Examples.

the ordering of elements given by a mark sequence is simply a topological ordering.

A DAG  $G$  is called *weighted* if there is an assignment of real numbers to each vertex of  $G$ . We call a topological ordering *non-negative* if the sum of the weights of the vertices in every prefix is non-negative. Similarly a mark-unmark (or mark) sequence is *non-negative* if at each step the sum of the weights in  $M_i$  is non-negative.

Clearly, a non-negative mark sequence is equivalent to a non-negative topological ordering. However, a non-negative mark-unmark sequence may give a negative topological ordering (we use *negative* in place of “not non-negative”). For example, let  $G$  be a weighted DAG on four vertices  $\{a, b, c, d\}$  with a single edge  $bc$  and weights  $w(a) = w(c) = w(d) = 1$ ,  $w(b) = -1$ . Consider the following non-negative mark-unmark sequence: mark  $a, b, c$ , then unmark  $a$ , then mark  $d$  and  $a$ . This gives the topological ordering  $b, c, d, a$ , which is negative. This suggests the following question of Erickson<sup>5</sup>: is there a weighted DAG  $G$  that has a non-negative mark-unmark sequence but no non-negative mark sequence?

We answer this question in the negative with the following theorem.

**Theorem 1.** *If a weighted DAG  $G$  has a non-negative mark-unmark sequence, then  $G$  also has a non-negative mark sequence.*

This problem was motivated by a question of Eppstein,<sup>6</sup> which asked to determine the complexity to decide whether a weighted DAG  $G$  has a non-negative topological ordering. His motivation was related to abstract Fréchet distance problems. This question turned out to be practically equivalent to an NP-complete problem of Garey and Johnson [1], called SEQUENCING TO MINIMIZE MAXIMUM CUMULATIVE COST, which we will not define here in its

full generality. In fact, the problem is NP-complete even in the following special case.

**Theorem 2.** *Let  $G$  be a weighted DAG such that every vertex is either a source or a sink. Deciding whether  $G$  has a non-negative topological ordering is NP-complete.*

The proof of hardness is through a series of reductions, which have been noticed/proved by different people (including the authors) and are hard to gather from the internet, so we include the full proof of Theorem 2 in Section 3. Theorem 1 is proved in Section 2.

## 2. Marking and unmarking

In this section we prove Theorem 1.<sup>7</sup> In particular, given a weighted DAG  $G$  and a non-negative mark-unmark sequence, we will construct a non-negative mark sequence for  $G$ . We begin with some definitions. By  $w(X)$  we denote the sum of the weights of the elements of a set of vertices  $X$ . We say that a set  $Y \subseteq X$  is *X-indirected* if every edge between  $Y$  and  $X \setminus Y$  is directed towards  $Y$ . Similarly, say that a set  $Y \subseteq X$  is *X-outdirected* if every edge between  $Y$  and  $X \setminus Y$  is directed towards  $X \setminus Y$ . For simplicity, we call a set of vertices  $Y$  of a DAG  $G$  *outdirected (indirected)* if  $Y$  is  $V(G)$ -outdirected ( $V(G)$ -indirected). Note that this definition corresponds to the definition of indirected given in the previous section. Fig. 1 (a) gives an example of  $X$ -indirected and  $X$ -outdirected sets.

**Proof of Theorem 1.** Let  $G$  be a weighted DAG with a non-negative mark-unmark sequence. Let  $M_1, M_2, \dots, M_t$  be a mark-unmark sequence with at least one unmark step (otherwise we are done) of minimum length. For  $i \in [t]$ ,<sup>8</sup> put  $U_i = (\cup_{j=1}^{i-1} M_j) \setminus M_i$ , i.e., the set of elements that were marked in the first  $i - 1$  steps, but later became unmarked (in one of the first  $i$  steps). Note that  $U_i$

<sup>5</sup> Positive topological ordering, take 2, <http://cstheory.stackexchange.com/questions/1399>.

<sup>6</sup> Positive topological ordering, <http://cstheory.stackexchange.com/questions/1346>.

<sup>7</sup> A sketch of this proof posted by the fourth author can be found online at <http://cstheory.stackexchange.com/questions/1399>.

<sup>8</sup> Here (and later)  $[t]$  stands for  $\{1, \dots, t\}$ .

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