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Topological orderings of weighted directed acyclic graphs

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ABSTRACT

We call a topological ordering of a weighted directed acyclic graph *non-negative* if the sum of weights on the vertices in any prefix of the ordering is non-negative. We investigate two processes for constructing non-negative topological orderings of weighted directed acyclic graphs. The first process is called a *mark sequence* and the second is a generalization called a *mark–unmark sequence*. We answer a question of Erickson by showing that every non-negative topological ordering that can be realized by a mark–unmark sequence can also be realized by a mark sequence. We also investigate the question of whether a given weighted directed acyclic graph has a non-negative topological ordering. We show that even in the simple case when every vertex is a source or a sink the question is NP-complete.

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1. Introduction

A *directed acyclic graph* (or DAG) is a directed graph with no directed cycles. A subset M of vertices of G is *outdirected* if every edge between M and $V(G) \setminus M$ is directed towards $V(G) \setminus M$ (i.e., edges directed towards M are contained in M). A *prefix of length* k of a sequence s is the subsequence of the first k terms of s. A *topological ordering* of a DAG G is an ordering of the vertices of G such that every prefix of the ordering is outdirected. The following two processes yield topological orderings of a given DAG G with n vertices.

A *mark sequence* of G is a sequence $M_1, M_2, \ldots, M_n$ of subsets of $V(G)$ formed in the following way: first choose an arbitrary source v and put $M_1 = \{v\}$, i.e., *mark* v in step 1. For $i = 2, 3, 4, \ldots, n$, choose a vertex $u \notin M_{i-1}$ such that $\{u\} \cup M_{i-1}$ is outdirected and put $M_i = \{u\} \cup M_{i-1}$, i.e., *mark* u in step i.

A *mark–unmark sequence* of G is a sequence of subsets of $V(G)$ formed in the following way: first choose an arbitrary source v and put $M_1 = \{v\}$, i.e., *mark* v in step 1. For $i = 2, 3, 4, \ldots, n$ either (i) choose a vertex $u \notin M_{i-1}$ such that $\{u\} \cup M_{i-1}$ is outdirected and put $M_i = \{u\} \cup M_{i-1}$, i.e., *mark* u in step i or (ii) choose a vertex $u \in M_{i-1}$ such that $M_{i-1} \setminus \{u\}$ is outdirected and put $M_i = M_{i-1} \setminus \{u\}$, i.e., *unmark* u in step i. This process stops when $M_i = V(G)$.

Clearly, mark–unmark sequences are a generalization of mark sequences. Because we only mark a vertex if the new set M_i is outdirected, we get a topological ordering by arranging the vertices of G by the last step in which they were marked in the mark–unmark sequence. In particular,

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(a) Y_{in} is an X-*indirected* set, Y_{out} is an X-*outdirected* set.

(b) 12 steps of a mark-unmark sequence, number j (resp. $-j$) denotes that the vertex was added (resp. removed) in step j.

Fig. 1. Examples.

the ordering of elements given by a mark sequence is simply a topological ordering.

A DAG G is called *weighted* if there is an assignment of real numbers to each vertex of G. We call a topological ordering *non-negative* if the sum of the weights of the vertices in every prefix is non-negative. Similarly a mark–unmark (or mark) sequence is *non-negative* if at each step the sum of the weights in M_i is non-negative.

Clearly, a non-negative mark sequence is equivalent to a non-negative topological ordering. However, a non-negative mark–unmark sequence may give a negative topological ordering (we use *negative* in place of "not non-negative"). For example, let G be a weighted DAG on four vertices $\{a, b, c, d\}$ with a single edge bc and weights $w(a) = w(c) = w(d) = 1$, $w(b) = -1$. Consider the following non-negative mark–unmark sequence: mark a, b, c, then unmark a, then mark d and a. This gives the topological ordering b, c, d, a, which is negative. This suggests the following question of Erickson[5]: is there a weighted DAG G that has a non-negative mark–unmark sequence but no non-negative mark sequence?

We answer this question in the negative with the following theorem.

Theorem 1. *If a weighted DAG G has a non-negative mark–unmark sequence, then G also has a non-negative mark sequence.*

This problem was motivated by a question of Eppstein,[6] which asked to determine the complexity to decide whether a weighted DAG G has a non-negative topological ordering. His motivation was related to abstract Fréchet distance problems. This question turned out to be practically equivalent to an NP-complete problem of Garey and Johnson [1], called SEQUENCING TO MINIMIZE MAXIMUM CUMULATIVE COST, which we will not define here in its full generality. In fact, the problem is NP-complete even in the following special case.

Theorem 2. *Let G be a weighted DAG such that every vertex is either a source or a sink. Deciding whether G has a non-negative topological ordering is* NP*-complete.*

The proof of hardness is through a series of reductions, which have been noticed/proved by different people (including the authors) and are hard to gather from the internet, so we include the full proof of Theorem 2 in Section 3. Theorem 1 is proved in Section 2.

2. Marking and unmarking

In this section we prove Theorem 1.[7] In particular, given a weighted DAG G and a non-negative mark–unmark sequence, we will construct a non-negative mark sequence for G. We begin with some definitions. By $w(X)$ we denote the sum of the weights of the elements of a set of vertices X. We say that a set $Y \subseteq X$ is X-*indirected* if every edge between Y and $X \setminus Y$ is directed towards Y. Similarly, say that a set $Y \subseteq X$ is X-*outdirected* if every edge between Y and $X \setminus Y$ is directed towards $X \setminus Y$. For simplicity, we call a set of vertices Y of a DAG G *outdirected* (*indirected*) if Y is $V(G)$-outdirected ($V(G)$-indirected). Note that this definition corresponds to the definition of indirected given in the previous section. Fig. 1 (a) gives an example of X-indirected and X-outdirected sets.

Proof of Theorem 1. Let G be a weighted DAG with a non-negative mark–unmark sequence. Let $M_1, M_2, \ldots, M_t$ be a mark–unmark sequence with at least one unmark step (otherwise we are done) of minimum length. For $i \in [t]$,[8] put $U_i = (\cup_{j=1}^{i-1} M_j) \setminus M_i$, i.e., the set of elements that were marked in the first $i-1$ steps, but later became unmarked (in one of the first i steps). Note that U_i

[5] Positive topological ordering, take 2, http://cstheory.stackexchange.com/questions/1399.

[6] Positive topological ordering, http://cstheory.stackexchange.com/questions/1346.

[7] A sketch of this proof posted by the fourth author can be found online at http://cstheory.stackexchange.com/questions/1399.

[8] Here (and later) $[t]$ stands for $\{1, \ldots, t\}$.

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