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Hamiltonian decomposition of generalized recursive circulant graphs ☆

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ABSTRACT

In 2012, Tang et al. [9] proposed a new class of graphs called generalized recursive circulant (GRC) graphs, which is an extension of recursive circulant graphs. GRC graphs have a more flexible structure than recursive circulant graphs, while retaining their attractive properties, such as degree, connectivity, diameter, and routing algorithm. In this paper, the Hamiltonian decomposition of some GRC graphs is discussed.

1. Introduction

An interconnection network can be modeled as a *graph*, in which vertices represent processors or computers and edges represent connections or communication links. Throughout this paper, a network is represented as a loopless undirected graph G. For the graph theoretic definition and notation, we follow [4]. A graph G is a two-tuple (V, E), where V is a nonempty set and E is a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of $V\}$. We say that V is the vertex set and E is the edge set. The *degree* of a vertex v, denoted by $\deg(v)$, is the number of edges incident with v. A graph G is called *k-regular* if $\deg(v) = k$ for every vertex $v \in V(G)$. Two vertices, u and v, of a graph G are *adjacent* if $(u, v) \in E(G)$. A *simple path* (abbreviated as *path*) is a sequence of adjacent edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{m-1}, v_m)$, written as $\langle v_0, v_1, v_2, \ldots, v_m\rangle$, in which all the vertices $v_0, v_1, v_2, \ldots, v_m$ are distinct except possibly $v_0 = v_m$. The path $\langle v_0, v_1, v_2, \ldots, v_m\rangle$ is called a *cycle* if $v_0 = v_m$. A cycle that traverses every vertex exactly once is called a *Hamiltonian cycle*. A k-regular graph G is *Hamiltonian decomposable* if its edge set can be partitioned into $\frac{k}{2}$ edge-disjoint Hamiltonian cycles when k is even or $\frac{k-1}{2}$ edge-disjoint Hamiltonian cycles and a perfect matching when k is odd. The presence of edge-disjoint cycles in a network enhances fault tolerance, efficiency routing, broadcasting, etc. [5,6]. Hamiltonian decomposition of some graph families, such as *complete k-uniform hypergraphs* [2], *5-star graphs* [6], and *rectangular twisted toruses* [7], is discussed extensively in the literature. Furthermore, Micheneau [8] and Biss [3] proved that *recursive circulant graphs* are Hamiltonian decomposable. The edge-pancyclicity and the super-connected property of recursive circulant graphs are studied by Araki [1] and Tsai et al. [10], respectively.

The *generalized recursive circulant (GRC) graph* [9] is an extension of the recursive circulant graph, proposed by Tang et al. in 2012. GRC graphs have a more flexible struc-

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ture than recursive circulant graphs in varying the number of vertices, and retain attractive properties of recursive circulant graphs, such as *degree*, *connectivity*, *diameter*, and *routing algorithm*. With the structural characteristics of GRC graphs, the embedding of a *mesh* is congestion-free and a *torus* is congestion-2 and dilation-2. Tang et al. conjectured that GRC graphs are Hamiltonian decomposable. In this paper, we focus on the Hamiltonian decomposition of some GRC graphs.

The definition of a generalized recursive circulant (GRC) graph is stated as follows.

Definition 1. (See [9].) A generalized recursive circulant (GRC) graph is denoted by $GR(h_k, h_{k-1}, \ldots, h_1)$, where $h_i \geq 2$ for $1 \leq i \leq k$. Each vertex in the graph can be labeled as a k-tuple $v_k v_{k-1} \ldots v_1$ with $0 \leq v_i \leq h_i - 1$ for $1 \leq i \leq k$, which is a mixed radix number representation. A vertex $v_k v_{k-1} \ldots v_1$ is adjacent to all of those vertices labeled by $v'_k \ldots v'_{i+1} v'_i v_{i-1} \ldots v_1$ for $1 \leq i \leq k$ if $v'_k \ldots v'_{i+1} v'_i$ is equal to $v_k \ldots v_{i+1} v_i \pm 1$, where the addition and subtraction in each dimension include carry and borrow.

The GRC graph $GR(h_k, h_{k-1}, \ldots, h_1)$ is $2k$-regular and $(2k-1)$-regular if $h_k \geq 3$ and $h_k = 2$, respectively. Given a $GR(h_k, h_{k-1}, \ldots, h_1)$ while $k \geq 2$, let V_i denote the vertex set $\{v_k v_{k-1} \ldots v_2 i \mid 0 \leq v_j \leq h_j - 1\}$ for $0 \leq i \leq h_1 - 1$. We use $GR_i(h_k, h_{k-1}, \ldots, h_1)$ to denote the subgraph of $GR(h_k, h_{k-1}, \ldots, h_1)$ induced by V_i. Therefore, $V(GR(h_k, h_{k-1}, \ldots, h_1)) = \cup_{i=0}^{h_1-1} V(GR_i(h_k, h_{k-1}, \ldots, h_1))$, and

$$V(GR_i(h_k, h_{k-1}, \ldots, h_1)) \cap V(GR_j(h_k, h_{k-1}, \ldots, h_1)) = \emptyset,$$

for any $i \neq j$. Each induced subgraph $GR_i(h_k, h_{k-1}, \ldots, h_1)$ is isomorphic to $GR(h_k, h_{k-1}, \ldots, h_2)$ for $0 \leq i \leq h_1 - 1$.

We also need some terms in $GR(h_k, h_{k-1}, \ldots, h_1)$. For $h_k = 2$, we define the edge set $X = \{(0v_{k-1}v_{k-2}\ldots v_1, 1v_{k-1}v_{k-2}\ldots v_1) \mid 0 \leq v_i \leq h_i - 1 \text{ for } 1 \leq i \leq k-1\}$, and $GR^*(h_k, h_{k-1}, \ldots, h_1) = GR(h_k, h_{k-1}, \ldots, h_1) - X$. Obviously, X is a perfect matching of $GR(h_k, h_{k-1}, \ldots, h_1)$. For $h_k \geq 3$, we define $GR^*(h_k, h_{k-1}, \ldots, h_1) = GR(h_k, h_{k-1}, \ldots, h_1)$. Moreover, in $GR(h_k, h_{k-1}, \ldots, h_1)$ (respectively, $GR^*(h_k, h_{k-1}, \ldots, h_1)$), a path is called *a successive m-path of* $GR(h_k, h_{k-1}, \ldots, h_1)$ (respectively, $GR^*(h_k, h_{k-1} \ldots, h_1)$) if and only if it is labeled as $\langle v_k v_{k-1} \ldots v_1, v_k v_{k-1} \ldots v_1 + 1, \ldots, v_k v_{k-1} \ldots v_1 + m\rangle$. Similarly, in $GR_i(h_k, h_{k-1}, \ldots, h_1)$ (respectively, $GR_i^*(h_k, h_{k-1}, \ldots, h_1)$), a path is called *a successive m-path of* $GR_i(h_k, h_{k-1}, \ldots, h_1)$ (respectively, $GR_i^*(h_k, h_{k-1}, \ldots, h_1)$) if and only if it is labeled as $\langle v_k v_{k-1} \ldots v_1, v_k v_{k-1} \ldots v_1 + h_1, \ldots, v_k v_{k-1} \ldots v_1 + m \times h_1\rangle$.

The relationship of circulant graphs, recursive circulant graphs, and GRC graphs is as the following:

$$\text{recursive circulant graphs} \subset \text{GRC graphs} \subset \text{circulant graphs}$$

The rest of this paper is organized as follows. In Section 2, Hamiltonian decomposition of GRC graphs is discussed. In Section 3, we give concluding remarks.

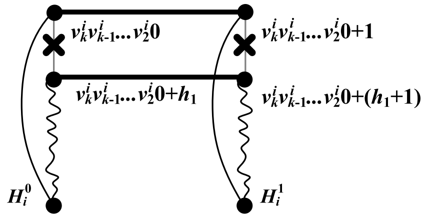

Fig. 1. $h_1 = 2$.

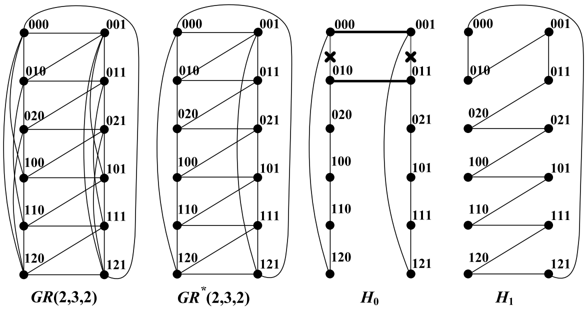

Fig. 2. Hamiltonian decomposition of $GR^*(2, 3, 2)$.

2. Hamiltonian decomposition of GRC graphs

Lemmas 1–4 provide some schemes for constructing edge-disjoint Hamiltonian cycles of GRC graphs recursively. Note that in Lemmas 2–4, the successive 1-paths and 2-paths of H_j^i (H_i, respectively) for all i and j are vertex-disjoint.

For $h_k = 2$, $GR(h_k)$ is a clique with two vertices and can be decomposed into a perfect matching. For $h_k \geq 3$, $GR(h_k)$ is a cycle of length h_k and is Hamiltonian decomposable such that every edge $(v_k, v_k + 1)$ for $0 \leq v_k \leq h_k - 1$ is on the Hamiltonian cycle. Thus, we have the following lemma.

Lemma 1. *If $h_k = 2$, then $GR(h_k)$ can be decomposed into a perfect matching. If $h_k \geq 3$, then $GR(h_k)$ is a Hamiltonian cycle, denoted by H_k, such that a successive 2-path $\langle v_k, v_k + 1, v_k + 2\rangle \in H_k$.*

Lemma 2. *Let $k \geq 2$ and $h_1 = 2$ in $GR^*(h_k, h_{k-1}, \ldots, h_1)$. For $0 \leq i \leq h_1 - 1$, assume that each $GR_i^*(h_k, h_{k-1}, \ldots, h_1)$ can be decomposed into k' ($k' = k-2$ if $h_k = 2$ and $k' = k-1$ if $h_k \geq 3$) disjoint Hamiltonian cycles, denoted by $H_0^i, H_1^i, \ldots, H_{k'-1}^i$, such that a successive 1-path $\langle v_k^i v_{k-1}^i \ldots v_2^i i, v_k^i v_{k-1}^i \ldots v_2^i i + h_1\rangle \in H_j^i$ for $0 \leq j \leq k' - 1$. Then, $GR^*(h_k, h_{k-1}, \ldots, h_1)$ can be decomposed into $k' + 1$ disjoint Hamiltonian cycles, denoted by $H_0, H_1, \ldots, H_{k'}$, such that a successive 1-path $\langle v_k v_{k-1} \ldots v_1, v_k v_{k-1} \ldots v_1 + 1\rangle \in H_i$ for $0 \leq i \leq k'$.*

Proof. For $0 \leq i \leq k' - 1$, let $H_i = H_i^0 \cup H_i^1 \cup \{(v_k^i v_{k-1}^i \ldots v_2^i 0, v_k^i v_{k-1}^i \ldots v_2^i 0 + 1), (v_k^i v_{k-1}^i \ldots v_2^i 0 + h_1, v_k^i v_{k-1}^i \ldots v_2^i 0 + (h_1 + 1))\} - \{(v_k^i v_{k-1}^i \ldots v_2^i 0, v_k^i v_{k-1}^i \ldots v_2^i 0 + h_1), (v_k^i v_{k-1}^i \ldots v_2^i 0 + 1, v_k^i v_{k-1}^i \ldots v_2^i 0 + (h_1 + 1))\}$. See Fig. 1 for illustration. Finally, let $H_{k'} = GR^*(h_k, h_{k-1}, \ldots, h_1) - \cup_{i=0}^{k'-1} H_i$. Therefore, $GR^*(h_k, h_{k-1}, \ldots, h_1)$ can be decom-