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A bounded space algorithm for online circle packing

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ABSTRACT

We study the Online Circle Packing Problem where we need to pack circles that arrive online in square bins with the objective to minimize the number of bins used. An online algorithm is said to have bounded space if at any given time, only a constant number of bins are open, circles are packed only in open bins and once a bin is closed it cannot be reopened. In particular, we present a 2.4394-competitive bounded space algorithm for this problem and a 2.2920 lower bound on the competitive ratio of any online bounded space algorithm for this problem.

1. Introduction

In the *Online Circle Packing Problem*, one has infinitely many square bins and receives a list of circles (given by its radii) in an online fashion. When a circle arrives, it must be packed in a bin, without intersecting other circles or the borders of the bin. Also, after packing the circle, it cannot be moved to another bin or another position in the bin. The objective is to minimize the number of bins used.

We say that an online algorithm A has an asymptotic competitive ratio of α if, for every instance I, $A(I) \le \alpha\,\mathrm{OPT}(I) + C$ where $A(I)$ is the value of the solution produced by algorithm A, $\mathrm{OPT}(I)$ is the value of an optimal offline solution and C is a constant. In this paper we present an online algorithm with asymptotic competitive ratio at most 2.4394. This algorithm has the nice property that it has *bounded space*, that is, at any time there is at most a constant number of open bins. After a bin is closed, it is not opened anymore and, hence, does not receive new circles. Also, we present a 2.2920 lower bound on the competitive ratio of any online bounded space algorithm for this problem.

Previous works The book of Szabo et al. [10] presents many results regarding finding the maximum common radius of k circles that can be packed in a unit square for several values of k along with other related problems. The website maintained by Specht [9] collects even more results, not only regarding the packing of circles in a unit square but also the packing of circles in a circle, in an isosceles right triangle, in a semicircle, in a circular quadrant and other problems. Some applications of circle packing includes obtaining a maximal coverage of radio towers in a geographical region [10] and construct photo collages [12]. A review on circle packing problems and methodologies can be found in [5].

For the offline circle packing problem, there is an asymptotic polynomial time approximation scheme by Miyazawa et al. [7] when we can augment the bin in one direction which can also be adapted to the circle strip packing problem. Note that, as shown by Demaine

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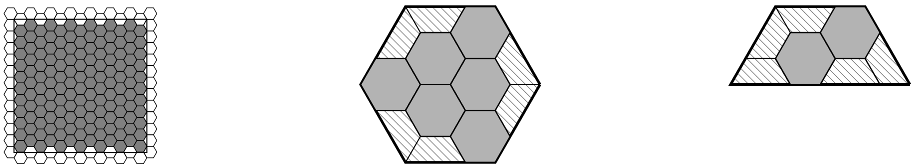

(a) bin (b) h-bin (c) t-bin

Fig. 1. In (a), a division of a bin in h-bins of type $(i, 0)$. In (b), a subdivision of an h-bin of type (i, k) and in (c) a subdivision of a t-bin of type (i, k) in h-bins and t-bins of type $(i, k+1)$ for $C = 3$.

et al. [1], it is NP-hard to decide if a set of circles can be packed into a square bin.

This online problem is already studied in the literature when the objects to be packed are squares, rectangles and hyperboxes. Epstein and Van Stee [4] developed a bounded space online algorithm for the d-dimensional online hypercube packing, extended this algorithm for the d-dimensional online hyperbox packing, for the variable-sized d-dimensional bin packing problem and for the online bin packing with resource augmentation. Later on, Epstein and Van Stee [3] presented numerical lower and upper bounds for d-dimensional online bounded space hypercube for $d \in \{2, \dots, 7\}$. Epstein [2] presented bounded and unbounded space algorithms for the two-dimensional online rectangle packing with orthogonal rotations.

2. An algorithm for Online Circle Packing

We start by presenting an algorithm for the Online Circle Packing Problem. For simplicity, we consider that a bin is a square of side length 1. The algorithm divides the circles into large circles and small circles. Given some positive integer constant M, a circle is said to be *large* if its radius is bigger than $1/M$ and, otherwise, it is said to be *small*.

For every positive integer i, let ρ_i^* be the largest value such that i circles of radius ρ_i^* can be packed in a bin. For example, $\rho_1^* = 0.5$ since we can pack a circle of radius 0.5 in a bin but we cannot pack a circle of radius $0.5 + \varepsilon$ in a bin for any $\varepsilon > 0$. As only some few values of ρ_i^* are currently known, we will use the best known lower bound on the value of unknown ρ_i^*, by ρ_i, obtained from the literature [9]. For algorithms that can compute ρ_i, we refer to [10]. Let K be such that $\rho_K > 1/M \geq \rho_{K+1}$. We will say that a large circle is of type $1 \leq i < K$ if its radius is at most ρ_i and larger than ρ_{i+1} and of type K if its radius is at most ρ_K and larger than $1/M$. We pack large circles of the same type together, packing at most i circles of type i in the same bin.

Let $C > 1$ be a positive integer constant that is a multiple of 3. We say that a small circle of radius r is of type i (for $M \leq i < CM$), subtype k (or, simply that r is of type (i, k)) if $1/(i+1) < C^k r \leq 1/i$ where k is the largest integer such that $C^k r \leq 1/M$. Small circles are packed using a recursive hexagonal packing defined later.

At a given time, the algorithm maintains at most K bins opened to pack large circles and $(C-1)M$ bins opened to pack small circles, and thus, it has bounded space. Recall that the area of a hexagon of side length ℓ is $3\sqrt{3}\ell^2/2$. Also, the radius of the inscribed circle of a hexagon of side length ℓ is $\sqrt{3}\ell/2$. That is, it is possible to pack a circle of radius r in a hexagon of side length $2r/\sqrt{3}$.

The algorithm generates three types of sub-bins. For $1 \leq i \leq K$, a *c-bin* of type i is a circular bin of radius ρ_i (it is used only for large circles). For $M \leq i < CM$ and $k \geq 0$, an *h-bin* of type (i, k) is a hexagonal bin of side $2/(\sqrt{3}C^k i)$ and a *t-bin* of type (i, k) is a trapezoidal bin created by cutting an h-bin of type (i, k) in half with a cut parallel to two of its sides. Notice that h-bins and t-bins are only used to pack small circles. Also, the algorithm will divide a bin into h-bins of type $(i, 0)$. This is done by selecting a hexagonal tiling of a bin where there is a hexagon on the left bottom part of the bin with two of its sides parallel to the bottom of the bin and by removing the hexagons that are not properly contained in the bin. See Fig. 1a. The algorithm also divides h-bins and t-bins in additional sub-bins. In Lemma 2.1, we show how this can be done without losing any area of the original sub-bin.

Lemma 2.1. *For $M \leq i < CM$ and $k \geq 0$, if C is a multiple of 3 then it is possible to partition an h-bin (or a t-bin) of type (i, k) in h-bins and t-bins of type $(i, k+1)$.*

Proof. Consider an h-bin of type (i, k) scaled so that its side length is C and embedded in the plane with its center at the origin, with two sides parallel to the x-axis. Notice that, after the scaling, an h-bin of type $(i, k+1)$ has side length 1. Finally, consider the hexagonal packing of the plane where the hexagons has side length 1 and have two sides parallel to the x-axis and, also, there is a hexagon with its leftmost point at $(0, 0)$.

Notice that, because C is a multiple of 3, we have that the leftmost point of the h-bin (at $(-C, 0)$) is the leftmost point of a hexagon. Thus, the segment which goes from the h-bins' leftmost point to the leftmost point of its top (at $(-C/2, \sqrt{3}C/2)$), which has an angle of 60 degrees, either cuts hexagons in half or cuts between hexagons. Again, as C is a multiple of 3, it cuts a hexagon exactly in half at leftmost point of the h-bin's top, ending at the hexagon rightmost point of its top. Now, by reflection symmetry over the x-axis, the same is true for the segment which goes from the h-bins' leftmost point to the leftmost point of its base (at $(-C/2, -\sqrt{3}C/2)$). Finally, by rotational symmetry over 120 degrees, the same is true for all edges of the h-bin. See Fig. 1b.

For a t-bin of type (i, k), the result follows from this observation along with the fact that we will also split the hexagons with y-coordinate 0 horizontally in half. See Fig. 1c. □