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Hamiltonicity of hypercubes with faulty vertices ☆

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ABSTRACT

Let Q_n denote the n-dimensional hypercube with the bipartition W and B. Assume $W_0 = \{a_1, a_2, \ldots, a_k\} \subset W$ and $B_0 = \{b_1, b_2, \ldots, b_k\} \subset B$, where $k \geq 1$ and $n \geq k+2$. The following is a long-standing conjecture (Locke, 2001 [15]): The graph $G = Q_n - (W_0 \cup B_0)$ contains a Hamiltonian cycle. Very recently, Locke' conjecture was proven when $k \leq 4$ [3] or $n \geq 2k+2$ [14]. In this paper, we obtain the following two results: (1) If a_i is adjacent to b_i for each i with $1 \leq i \leq k$, then Locke' conjecture holds (2) Moreover, if $n \geq k+3$, then the graph G is Hamilton laceable, i.e., for any $a \in W \backslash W_0$ and $b \in B \backslash B_0$, the graph G contains a Hamiltonian path connecting a and b, and the lower bound $k+3$ is optimal. Our results may find some applications.

1. Introduction

It is well known that the n-dimensional hypercube or n-cube, denoted by Q_n, is one of the most popular and efficient interconnection networks. It possesses many excellent properties such as recursive structure, symmetry, small diameter, low degree, popular topological structure embedding, and easy routing. The Hamiltonian property is one of the major requirements in designing network topologies since a topology structure containing Hamiltonian paths or cycles can efficiently simulate many algorithms designed on linear arrays or rings. It is well known that the n-cube contains a Hamiltonian cycle.

Element (edge and/or vertex) failure is inevitable when a large parallel computer system is put in use. Therefore, fault tolerant capacity of a network is a critical issue in parallel computing. There is a large amount of literature on cycle and/or path embedding into interconnection networks with or without faulty elements, see a survey [22] and references therein. For (fault-tolerant) path or cycle embedding of hypercubes, also see references [2–14, 17–21].

In 2001, Locke [15] considered Hamiltonicity of hypercubes with faulty vertices and proposed the following conjecture that remains open.

Conjecture. *The n-cube with deleting k vertices from each bipartite set contains a Hamiltonian cycle if $n \geq k+2$.*

In [16] Locke et al. proved the conjecture for $k = 1$. Very recently, the conjecture was proved when $k \leq 4$ [3] or $n \geq 2k+2$ [14].

In this paper, we obtain the following results relative to the conjecture.

Theorem 1. *Let Q_n be the n-cube with bipartition W and B, and let V be a set of all the endvertices of k independent edges in Q_n, where $k \geq 1$ and $n \geq k+2$. Then the graph $G = Q_n - V$ contains a Hamiltonian cycle. Moreover, if $n \geq k+3$, then the graph G is Hamilton laceable, i.e., for any two vertices $a \in W \backslash V$ and $b \in B \backslash V$, the graph G contains a Hamiltonian path connecting a and b, and the lower bound $k+3$ is optimal.*

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2. Preliminaries

The terminology and notation used in this paper mostly follow [1]. A graph $G = (V, E)$ means a simple graph, where $V = V(G)$ is its vertex-set and $E = E(G)$ is its edge-set. In a graph $k(\geq 2)$ edges are said to be independent if any two edges have no common end-vertex. We use $P = (v_0, v_1, \ldots, v_k)$ to denote a path with $k+1$ vertices $v_0, v_1, \ldots, v_k$ and k edges (v_i, v_{i+1}) for $i = 0, 1, \ldots, k-1$, where two vertices v_0 and v_k are called its end-vertices. If $k \geq 3$, $P = (v_1, \ldots, v_k)$ is a path and (v_1, v_k) is also an edge, then $C = P \cup (v_1, v_k)$ denotes a cycle. A cycle (respectively, path) containing all vertices of a graph G is called a Hamiltonian cycle (respectively, Hamiltonian path) of G. A graph G is called Hamiltonian if it contains a Hamiltonian cycle. A bipartite graph G is called Hamilton laceable if it contains a Hamiltonian path connecting any vertex from each partite set. It is clear that Hamiltonian laceability implies Hamiltonicity.

Let G be a graph, $E' \subset E(G)$ and $V' \subset V(G)$. By $G - E'$ we denote the graph obtained from G by deleting all edges in E', and by $G - V'$ we denote the graph obtained from G by deleting all vertices in V' and all edges incident with vertices in V'. If P and Q are two paths with only one common vertex being an end-vertex of both P and Q, then the notation $P \cup Q$ denotes the path induced by $E(P) \cup E(Q)$. If P and Q are two paths such that u and v are two end-vertices of both P and Q, and $V(P) \cap V(Q) = \{u, v\}$, then the notation $P \cup Q$ denotes the cycle induced by $E(P) \cup E(Q)$.

The n-cube is a graph with 2^n vertices, and its any vertex v is denoted by a unique n-bit binary string $v = \delta_n\delta_{n-1}\ldots\delta_2\delta_1$, where $\delta_i \in \{0, 1\}$ for $i = 1, 2, \ldots, n$. Two vertices of Q_n are adjacent if and only if their binary strings differ in exact one bit position. It is easy to show that Q_n is an n-regular bipartite graph. Let B and W denote the two partite sets of Q_n. Assume that (u, v) is an edge of Q_n. If the two binary strings of u and v differ in the i-th bit position, then the edge (u, v) is called an edge of dimension i in Q_n. The set of all edges of dimension i in Q_n is denoted by E_i. For any given h with $1 \leq h \leq n$, let $Q^{0,h}_{n-1}$ and $Q^{1,h}_{n-1}$ be two $(n-1)$-dimensional subcubes of Q_n induced by all vertices with the h-th bit being 0 and 1, respectively. By E_h Q_n is partitioned into $Q^{0,h}_{n-1}$ and $Q^{1,h}_{n-1}$, denoted by $Q_n - E_h = Q^{0,h}_{n-1} \bigcup Q^{1,h}_{n-1}$. For a given $\delta \in \{0, 1\}$, if v is a vertex of $Q^{\delta,h}_{n-1}$, then there is exactly one corresponding vertex in $Q^{1-\delta,h}_{n-1}$, denoted by v', such that $(v, v') \in E_h$.

The following results are well known. Ref. [9] introduced the-2-path bipanconnectivity and generalized the results.

Lemma 1. *Let s_1, t_1, s_2, t_2 be four vertices in the n-cube Q_n such that $s_i \in B$ and $t_i \in W$ for $i = 1, 2$.*

(1) (See [10].) If $n \geq 2$, then Q_n contains two vertex-disjoint paths P_1 and P_2, where P_i connects s_i and t_i for $i = 1, 2$, such that $V(P_1) \cup V(P_2) = V(Q_n)$. As a corollary, Q_n contains a Hamiltonian path connecting s_1 and t_1.

(2) (See [2].) If $n \geq 4$, then Q_n contains two vertex-disjoint paths P_3 and P_4, where P_3 connects s_1 and s_2, and P_4 connects t_1 and t_2, such that $V(P_3) \cup V(P_4) = V(Q_n)$.

3. The proof of Theorem 1

Let $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$ be k independent edges in Q_n, where $W_0 = \{a_1, a_2, \ldots, a_k\} \subset W$ and $B_0 = \{b_1, b_2, \ldots, b_k\} \subset B$.

We prove Theorem 1 by induction on n. If $3 = n \geq k+2$, then $k = 1$. It is clear that $Q_3 - \{a_1, b_1\}$ is Hamiltonian. If $4 = n \geq k+3$, then $k = 1$. One can check (or applying the result in [14]) that $Q_4 - \{a_1, b_1\}$ is Hamilton laceable and Hamiltonian. If $4 = n = k+2$, then $k = 2$. One can check (or applying Lemma 3.8 in [3]) that $Q_4 - \{a_1, b_1, a_2, b_2\}$ is Hamiltonian. Assume the theorem holds for $n-1(\geq 4)$. We shall show that the theorem holds for $n(\geq 5)$.

First, assume $n \geq k+2$. We shall show that the graph $Q_n - (W_0 \cup B_0)$ has a Hamiltonian cycle.

Since (a_i, b_i) is an edge for each i with $1 \leq i \leq k(< n)$, it is easy to see that there exists some h with $1 \leq h \leq n$ such that the h-th bit of a_i is the same as b_i for each i, that is, $(a_i, b_i) \notin E_h$ for each i with $1 \leq i \leq k$. Let $Q_n - E_h = Q^{0,h}_{n-1} \bigcup Q^{1,h}_{n-1}$, $L = Q^{0,h}_{n-1}$ and $R = Q^{1,h}_{n-1}$. Then either $(a_i, b_i) \in E(L)$ or $(a_i, b_i) \in E(R)$ for each i with $1 \leq i \leq k$. If $v \in V(L)$ (or $v \in V(R)$), then v' denotes the corresponding vertex of v in R (or in L). There are two cases to consider.

Case 1. $(W_0 \cup B_0) \subset V(L)$. (If $(W_0 \cup B_0) \subset V(R)$, the proof is similar.)

Since $n-1 \geq (k-1)+2$ and L is an $(n-1)$-cube, by induction hypothesis, the graph $L - \{a_1, b_1, \ldots, a_{k-1}, b_{k-1}\}$ contains a Hamiltonian cycle C_0. Then $a_k \in V(C_0)$ and $b_k \in V(C_0)$. There are two subcases.

Subcase 1.1. $(a_k, b_k) \in E(C_0)$.

Let $(a, a_k) \in E(C_0)$ and $(b, b_k) \in E(C_0)$ with $a \neq b_k$ and $b \neq a_k$. And let $(a, a') \in E_h$ and $(b, b') \in E_h$, where $a' \in V(R) \cap W$ and $b' \in V(R) \cap B$. It is clear that $P_0 = C_0 - \{a_k, b_k\}$ is a Hamiltonian path connecting a and b in the graph $L - (W_0 \cup B_0)$. By Lemma 1(1), the $(n-1)$-cube R contains a Hamiltonian path P_1 connecting a' and b'. Let $C = P_0 \cup (a, a') \cup (b, b') \cup P_1$. Then C is a Hamiltonian cycle in the graph $Q_n - (W_0 \cup B_0)$.

Subcase 1.2. $(a_k, b_k) \notin E(C_0)$.

Then the cycle C_0 contains four edges (x_1, a_k), (x_2, a_k), (y_1, b_k), (y_2, b_k) such that $x_i \neq b_k$ and $y_i \neq a_k$ for $i = 1, 2$. Let $(x_1, x'_1), (x_2, x'_2), (y_1, y'_1), (y_2, y'_2) \in E_h$, where $x'_1, x'_2 \in V(R) \cap W$ and $y'_1, y'_2 \in V(R) \cap B$. It is clear that the graph $C_0 - \{a_k, b_k\}$ is composed of two vertex-disjoint paths, denoted by P_1 and P_2. Since $n-1 \geq 4$ and R is an $(n-1)$-cube, by Lemma 1(2), R contains two vertex-disjoint paths P_3 and P_4, where P_3 connects x'_1 and x'_2 and P_4 connects y'_1 and y'_2 with $V(P_3) \cup V(P_4) = V(R)$. Let $C = P_1 \cup P_2 \cup P_3 \cup P_4 \cup (x_1, x'_1) \cup (x_2, x'_2) \cup (y_1, y'_1) \cup (y_2, y'_2)$. Then C is a Hamiltonian cycle in the graph $Q_n - (W_0 \cup B_0)$.

Case 2. There is some r with $1 \leq r \leq k-1$ such that $V_0 = \{a_1, b_1, \ldots, a_r, b_r\} \subset V(L)$ and $V_1 = \{a_{r+1}, b_{r+1}, \ldots, a_k, b_k\} \subset V(R)$.