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Tree-core and tree-coritivity of graphs

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ABSTRACT

A new graph parameter, called tree-coritivity, is introduced in this paper. A decycling-cut is a vertex-cut whose removal results in an acyclic graph. Let $\omega(G)$ be the number of connected components of a graph G. The tree-coritivity of a graph G is the maximum value of $\omega(G - S^*) - |S^*|$, where S^* takes over all decycling-cuts of G. It is shown that this parameter can be used to measure the vulnerability of a graph. We prove that the problem of computing the tree-coritivity of a graph is **NP**-complete. Moreover, we figure out the bounds of tree-coritivity of graphs, give a way to compute the tree-coritivity of the join graph, and determine the exact value of tree-coritivities for some special classes of graphs.

1. Introduction

The resistance of a graph (or a network) to operations, such as deletion of vertices or edges is one important property to reflect the reliability and efficiency of the graph. Thus far, there have appeared many literatures that investigated the vulnerability of a graph [16,3,10,8]. Generally, in an analysis of the vulnerability of a graph (or a network) to disruption, two important quantities (there may be others) are (1) the number of elements that are not functioning, (2) the number of remaining connected subnetworks. Based on the quantities, many graph parameters have been proposed for measuring the vulnerability of graphs. The vertex connectivity (or edge connectivity) is probably the earliest and also the most studied parameter [15,4,17,18,9]. However, in the early 1970s it was found that the connectivity only partly reflects the ability of graphs to retain certain degrees of connectedness after operations such as the removal of vertices or edges [1]. Therefore, a number of new parameters were introduced and studied, such as toughness [11,6,21], scattering number [19] (or coritivity [22]), integrity [1], tenacity [13,12], feedback number [14,7] (or decycling number [2]), and rupture degree [20]. In contrast to vertex (edge) connectivity, these measures consider not only the difficulty to break down the network but also the damage that has been caused.

Of all these parameters, we are more interested in the coritivity for its lively definitions and wide applications in real world, such as the problems in reliable communication networks and neural networks together with social psychology [23]. However, we still see much room for improving this parameter. First, coritivity involves only quantities (1) and (2). Nevertheless, in addition to these two quantities, there are also other quantities that may influence the vulnerability of a graph, e.g. (3) the number of cycles and their distribution in a graph. Naturally, coritivity shows its limitation in measuring the vulnerability of graphs due to missing quantity (3). Indeed, there exist many graphs that their connectedness is apparently different but they have the same coritivity. In other words, coritivity is incapable of distinguishing the connectedness of some graphs with

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different structures. Second, coritivity is defined only for connected graphs, and thus graphs that are unconnected cannot be measured by it.

Based on the above two aspects, a more appropriate parameter refining coritivity is necessary. Such a parameter should not only involve the quantity (3) besides (1) and (2), but also be feasible for general graphs including unconnected ones. In this paper, we introduce such a parameter called tree-coritivity and show that tree-coritivity enables us to distinguish the connectedness of some graphs, which cannot be distinguished by coritivity. Moreover, we figure out the bounds of tree-coritivity of graphs, give a way to compute the tree-coritivity of the join graph by characterizing the relation between the tree-coritivities of any two graphs and the join of them, and determine the exact value of tree-coritivities for some special classes of graphs.

Another contribution of this work is the analysis of the complexity of computing the tree-coritivity of a graph. Our result shows that this problem is NP-complete. Since the complexity for determining the coritivity of graphs has not been studied yet, our result on tree-coritivity may shed a light on the complexity of former.

2. Preliminaries

In what follow, we only consider undirected graphs without multiple edges or loops. For a graph G, we denote by $V(G)$ and $E(G)$ the set of vertices and edges of G, respectively. For a vertex $v \in V(G)$, $N_G(v)$ is the set of all vertices adjacent to v, and $d_G(v) = |N_G(v)|$ is the degree of v. The *maximum degree* and *minimum degree* of G is denoted by $\Delta(G)$ and $\delta(G)$, respectively. For brevity, $V(G), E(G), \Delta(G), \delta(G)$ are simplified by V, E, Δ, δ, respectively. Let H be a subgraph of G, we say H is a *spanning subgraph* of G if $V(H) = V(G)$. For a subset V' of $V(G)$, we denote by $E(V')$ the set of all edges of G with both endvertices in V'. The graph $G[V']$, whose vertex set is V' and edge set is $E(V')$, is the subgraph of G *induced* by V'. The graph $G[V - V']$ is simply written as $G - V'$. Analogously, for $E' \subseteq E$, the graph $G[E']$, whose vertex set is the set of all the endvertices of E' and edge set is E', is the subgraph of G *induced* by E'. The graph $G + E'$ is the graph obtained by adding the edge set E' to $E(G)$. For a graph G, we denote by $\omega(G)$ the number of connected components of G. Please refer to [5] for more notations of graph theory.

Recall that a vertex cut of a graph G is a $V' \subseteq V(G)$ such that $G - V'$ is disconnected. We use $\mathcal{C}(G)$ to denote the set of all the vertex cuts of G. A vertex cut S is said to be a *k-cut* if $|S| = k$. If G is not a complete graph, the connectivity $\kappa(G)$ of G is the minimum value of k over all k-cuts of G. Otherwise, we define the connectivity of a complete graph of order n to be $n-1$. Thus, $\kappa(G) = 0$ if G is either trivial or disconnected.

A *feedback vertex set* of a graph G is a set of vertices V' whose removal leaves a graph without cycles. However, in order to measure the vulnerability of G, we hope that V' is vertex cut, i.e. $G - V'$ is disconnected. We refer to a feedback vertex set V' as a *decycling-cut* of G if V' is a vertex cut, and denote by $\mathcal{T}(G)$ the set of all the decycling-cuts of G.

Definition 1 *(Core and coritivity).* Let G be a connected graph, and $n \geq 4$. The coritivity of G is defined as

$$h(G) = \max\{\omega(G - S) - |S| : S \in \mathcal{C}(G)\},$$

where $\omega(G - S)$ is the number of connected components of G. A set of vertices S^* with $h(G) = \omega(G - S^*) - |S^*|$ is said to be a *core* of G.

In view of the consideration that the number of cycles and their distribution in a graph can strongly affect the connectedness of the graph, we naturally introduce the concepts of *tree-core* and *tree-coritivity*, which generalize the concept of *core* and *coritivity*.

Definition 2 *(Tree-core and tree-coritivity).* Let G be a graph with at least one pair of distinct nonadjacent vertices. The tree-coritivity of G is defined as

$$h_t(G) = \max\{\omega(G - S) - |S| : S \in \mathcal{T}(G)\}.$$

A set of vertices S_t^* with $h_t(G) = \omega(G - S_t^*) - |S_t^*|$ is said to be a *tree-core* of G. In addition, since K_n contains no vertex cut, we define that the tree-coritivity of K_n is $2 - n$, and each subset with $n - 1$ vertices of $V(K_n)$ is a tree-core of K_n.

By the definition of tree-coritivity, it is not difficult to check that $h_t(P_m) = 1$ and $h_t(C_n) = 0$ for $m \geq 3$ and $n \geq 4$; Let G be a graph on at least 3 vertices. Then G contains a cycle if $h_t(G) \leq 0$; For any connected graph G with order $n(\geq 4)$, it has $h_t(G) \leq h(G)$.

As we analysed in Introduction, tree-coritivity can be used to further measure the vulnerability of graphs with the same coritivity. To see this, it is sufficient to show the following observation.

Observation. There are graphs which share the same coritivity but have distinct tree-coritivity.

We prove this observation by constructing a concrete example: Let $P_n = v_1 v_2 \cdots v_n$ be a path on n vertices; Add a vertex u_1 on one side of P_n, and connect u_1 with every v_i for $i = 1, 2, \cdots, n$; And add another vertex u_2 on the other side of P_n, and connect u_1 with each v_i for $i = 1, 2, \cdots, n-1$. The resulting graph is denoted by P_n'; Fig. 1 shows the graph P_5'.

For an integer $n \geq 4$, let C_n be a cycle on n vertices and P_n' be the graph defined above. Then, it is easy to see that $h(C_n) = h(P_n') = 0$, while $h_t(C_n) = 0$ and $h_t(P_n') = -1$. In addition, these two graphs apparently have different connectedness according to their structures. Thus, for these graphs, it can be seen that tree-coritivity is a better parameter than coritivity for distinguishing their connectedness.

We first introduce two simple and useful results as follows.

Lemma 2.1. *Let H be a spanning subgraph of G. Then $h_t(G) \leq h_t(H)$.*