Information Processing Letters 115 (2015) 760–764

Contents lists available at ScienceDirect

Information Processing Letters

www.elsevier.com/locate/ipl

Approximating the optimal sequence of acquisitions and sales with a capped budget

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ARTICLE INFO

Article history:
Received 22 October 2014
Received in revised form 5 May 2015
Accepted 5 May 2015
Available online 8 May 2015
Communicated by M. Chrobak

Keywords:
Approximation
Greedy algorithms
Knapsack problem

ABSTRACT

We introduce and study the approximability of the following problem. There is a set of useful objects that are available for purchase, and another set of useless objects that can be sold. Selling useless objects generates revenue which allows to acquire useful objects. We search for a sequence of decisions (buying or selling objects) which optimizes either the number of purchased objects or their global utility. One of the constraints is that, at any time, only a limited amount of money can be held.

1. Introduction

We consider a general problem in which an activity depends on a divisible and nonrenewable resource that is stored in a place with limited capacity. Some activities increase the quantity of resource, some others decrease this amount. An application of the model can be the operation of a mobile electric device having a battery with limited capacity. Two types of tasks exist: those whose execution requires a given amount of energy and those which increase the energy available.

Along this paper, we use the following abstract interpretation. There is a set of useful objects that are available for purchase, and another set of useless objects that can be sold. The sale of useless objects generates revenue which allows to acquire useful objects. Thus, money is the divisible resource in this setting, and only a limited amount can be collected at a time. We search for a sequence of decisions (buying or selling objects) which optimizes the number (or utility) of the purchased objects.

We start by giving a formal definition of the model in Section 2. Two optimization problems are derived, denoted by Π^c and Π^u, respectively. This paper deals with approximation algorithms [1,2]. Given $\rho \in (0, 1]$ and a maximization problem, we say that a solution with value *VAL* is a ρ-approximation of the optimum value *OPT* if $VAL \geq \rho\, OPT$. A ρ-approximation algorithm for a given problem returns a ρ-approximation for any instance. Moreover, its time complexity must be polynomial in the size of its input. In Section 3, we prove that both Π^c and Π^u cannot be approximated within the ratio $1/2 + \varepsilon$, unless $\mathbf{P} = \mathbf{NP}$. A 1/2-approximation algorithm for Π^c is given in Section 4. Section 5 is devoted to a $(1/2 - \gamma)$-approximation algorithm for Π^u.

2. General model, notations and problems

There is an agent and two sets of objects X and Y. Every object $z \in X \cup Y$ has a price $p(z) \in \mathbb{N}$ and a utility $u(z) \in \mathbb{N}$. Given a subset of objects Z, $p(Z)$ and $u(Z)$ stand for $\sum_{z \in Z} p(z)$ and $\sum_{z \in Z} u(z)$, respectively. X is the set of objects that the agent owns. We have $p(x) > 0$ and $u(x) = 0$ for every $x \in X$. Y is the set of objects that the agent would like to acquire. We have $p(y) \geq 0$ and $u(y) > 0$ for every $y \in Y$. Therefore, the agent can sell

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http://dx.doi.org/10.1016/j.ipl.2015.05.001
0020-0190/

$x \in X$ at a price of $p(x)$ or buy $y \in Y$, at a price of $p(y)$. The agent has a budget that changes over time. The budget at time t, denoted by $B(t)$, is a member of $\mathbb{N}$ and we suppose that $B(0) = B_0$ for some B_0 given in the input. The budget is capped, meaning that it cannot exceed a value B_{max} given in the input. The problem consists of selling and buying objects over time, under the following constraints.

- the agent can sell $x \in X$ at time t if x was not sold earlier. The budget increases by $\min\{p(x), B_{max} - B(t)\}$ where $B(t)$ designates the value of the budget just before x is sold. Having an upper bound on the budget means that a part of $p(x)$ may be wasted when $p(x) > B_{max} - B(t)$;
- the agent can buy $y \in Y$ at time t if y was not bought earlier and $B(t) \geq p(y)$. The budget is immediately decreased by $p(y)$. The utility increases by $u(y)$.

In the model, the changes of budget over time are only due to the buying and selling of objects. An object cannot be partially sold or bought. An object $y \in Y$ cannot be sold after it is bought. We assume that $p(z) \leq B_{max}$ for all object $z \in X \cup Y$ because no object of price strictly larger than B_{max} can be bought, and the budget is increased by at most B_{max} when an object is sold.

A solution s is an ordered list of objects. We can see s as a list of operations because at time t, the t-th object of s is sold or bought, depending on whether it belongs to X or Y.

Let $X(\mathsf{s})$ and $Y(\mathsf{s})$ be the subsets of X and Y which appear in s, respectively. We consider two problems related to the general model: find a solution s such that $|Y(\mathsf{s})|$ or $u(Y(\mathsf{s}))$ is maximum. The problems are called Π^c and Π^u (c and u stand for cardinality and utility, respectively). Maximizing $u(Y(\mathsf{s}))$ is equivalent to maximizing the utility of the objects that the agent acquires because $u(x) = 0$ for every $x \in X$.

Let us give a small example with $X = \{x_1\}$, $Y = \{y_1, y_2, y_3\}$, $B_0 = 2$ and $B_{max} = 6$. Utilities and prices are given in the next table.

	x_1	y_1	y_2	y_3
price	6	1	3	4
utility	0	2	5	3

A solution can be $\langle y_1, x_1, y_2 \rangle$. The budget is 2 initially. At time 1, it drops to 1 after y_1 is bought. At time 2 it becomes 6 after x_1 is sold. It is 3 once y_2 is bought at time 3. The global utility of the solution is 7.

2.1. Related work

Our problem resembles the well known KNAPSACK problem [1,3]. In this problem, there is a set of items, each having a size and a utility, and a given capacity. One seeks a subset of items whose total size does not exceed the capacity while the total utility is maximized. The main difference between KNAPSACK and our problem is that the capacity is fixed in the former whereas it can be modified over time in the latter. There are many variants of KNAPSACK (see Chapter 13 of [3] for a nonexhaustive list) but, up to our knowledge, none of them is equivalent to (or more general than) Π^u. The variant of KNAPSACK that is closest to Π^u (but not equivalent) is probably the MULTIPERIOD KNAPSACK [4]. There are m periods and the capacity of the knapsack is d_i during period i. Given $\{d_i : i = 1, \dots, m\}$ such that $d_1 \leq \dots \leq d_m$, the problem is to find a set of items, together with the period during which they are picked, such that the capacity in each period is not exceeded, and the total utility is maximized. Without the upper bound B_{max} on the budget, Π^u would be equivalent to KNAPSACK because it suffices to sell all the objects of X at the beginning and seek $Y' \subseteq Y$ such that $u(Y')$ is maximized under the constraint $\sum_{y \in Y'} p(y) \leq B_0 + \sum_{x \in X} p(x)$.

The problems studied in this article are also related to some scheduling problems. For example, Carlier and Rinnooy Kan study a scheduling problem subject to a resource constraint [5]. Apart from precedence constraints, executing a job requires that a given amount of some nonrenewable resource is available. In their model, at some given moments, the quantity of the resource is increased by a predefined amount. The difference with our model is that we can decide when the quantity of the resource is increased. Moreover, we suppose in this article that there exists an upper bound on the quantity of resource available at a time.

One can also mention similarities between Π^u and the RELOCATION problem [6,7]. Some buildings have to be redeveloped, i.e. the number of households that they can accommodate may decrease or increase. The problem is to find a redevelopment schedule with minimum duration such that every tenant is relocated.

3. Complexity

Given a value U, the decision version of Π^c (resp. Π^u) asks whether a feasible solution s satisfying $|Y(\mathsf{s})| \geq U$ (resp. $u(Y(\mathsf{s})) \geq U$) exists.

Theorem 1. *For any $\epsilon > 0$, Π^c is not $\left(\frac{1}{2} + \epsilon\right)$-approximable, unless* $\mathbf{P} = \mathbf{NP}$.

Proof. We conduct a gap reduction of PARTITION which is **NP**-complete [8]. Given r integers $\{\alpha_1, \dots, \alpha_r\}$ such that $\sum_{j=1}^{r} \alpha_j = 2B$ and $0 < \alpha_j < B$, PARTITION asks whether a subset $J \subseteq \{1, \dots, r\}$ such that $\sum_{j \in J} \alpha_j = B = \sum_{j \notin J} \alpha_j$ exists.

Given an instance I of PARTITION, build an instance I' of Π^c as follows: $B_0 = 0$, $B_{max} = B$, $X = \{x_1, \dots, x_r\}$ with $p(x_i) = \alpha_i$ for $i \in [r]$, $Y = \{y_1, y_2\}$ with $p(y_1) = p(y_2) = B$ and $u(y_1) = u(y_2) = 1$. We claim that I' admits a feasible solution s satisfying $|Y(\mathsf{s})| = 2$ if and only if I admits a partition.

If I admits a partition J, then sell the objects x_i with $i \in J$, buy y_1, sell the objects x_i with $i \notin J$ and buy y_2. Conversely, let s be a feasible solution such that $Y(\mathsf{s}) = \{y_1, y_2\}$ and y_1 is bought before y_2. Let J' be the indices of the objects in X sold before y_1 is bought. It must be $p(\{x_i : i \in J'\}) \geq p(y_1) = B$. The budget is 0 just after y_1 is purchased because $B_{max} = B$. Let J'' be the indices of the objects of X sold between the purchase of y_1 and y_2. It