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Incidence coloring of Cartesian product graphs ☆

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ABSTRACT

For a vertex $v \in V(G)$, the incidence neighborhood of v, denoted by $\mathrm{IN}(v)$, is the set $\{(v,e) : e \in E(G)$ and v is incident with $e\} \cup \{(u,e) : e = vu \in E(G)\}$. Let $S_\sigma(v)$ denote the set of colors assigned to $\mathrm{IN}(v)$ in G under incidence coloring σ and $s(\sigma) = \max\{|S_\sigma(v)| : v \in V(G)\}$. Let $G_1 \square G_2$ denote the Cartesian product of graphs G_1 and G_2. Let σ_i be an incidence coloring of graph G_i and $n(\sigma_i)$ the number of colors used by σ_i for $i \in \{1,2\}$. In this paper, we show that if $n(\sigma_1) \geqslant n(\sigma_2) - s(\sigma_2)$, then there exists an incidence coloring of $G_1 \square G_2$ which uses $n(\sigma_1) + s(\sigma_2)$ colors; otherwise, there exists an incidence coloring of $G_1 \square G_2$ using $n(\sigma_2)$ colors. Based on the result above, we settle the following conjecture in affirmative: For integer $p \geqslant 1$,

$$\chi_i(Q_n) = \begin{cases} n+1 & \text{if } n = 2^p - 1 \\ n+2 & \text{otherwise,} \end{cases}$$

where Q_n is the n-dimensional hypercube and $\chi_i(Q_n)$ is the incidence coloring number of Q_n.

1. Introduction

The *incidence set* of a graph $G = (V, E)$ is defined as $I(G) = \{(v,e) : v \in V(G), e \in E(G), v$ is incident with $e\}$, where $V(G)$ and $E(G)$ are the vertex and edge, respectively, sets of G. For a vertex $v \in V(G)$, the *incidence neighborhood* of v, denoted by $\mathrm{IN}(v)$, is the set $\{(v,e) : e \in E(G)$ and v is incident with $e\} \cup \{(u,e) : e = vu \in E(G)\}$. The incidences in the former set are the *near-incidences* of vertex v and the incidences in the latter set are *far-incidences* of vertex v. Two incidences (v_1, e_1) and (v_2, e_2) are *adjacent* if one of the following conditions holds: (i) $v_1 = v_2$, (ii) $e_1 = e_2$, or (iii) either v_2 is the other endpoint of e_1 or v_1 is the other endpoint of e_2. An *incidence coloring* σ of G is a mapping from $I(G)$ to a color set $\{1, \dots, n(\sigma)\}$ such that all adjacent incidences of G are assigned different colors, where $n(\sigma)$ denotes the number of colors used by σ. The *incidence coloring number* of G, denoted by $\chi_i(G)$, is the smallest number k such that G admits an incidence coloring σ with $n(\sigma) = k$.

The incidence coloring problem was introduced by Brualdi and Massey in [4]. They also conjectured that any graph G can be incidence-colored by $\Delta(G) + 2$ colors, where $\Delta(G)$ denotes the maximum degree of G. However, their conjecture was disproved by Guiduli [6]. In [6], Guiduli also showed that the incidence coloring problem is a special case of *directed star arboricity* which was introduced by Algor and Alon [1,2]. Note that the directed star arboricity problem has applications in the WDM (Wavelength Division Multiplexing) of a star optical network [3].

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For graphs G_1 and G_2, the *Cartesian product* $G_1 \square G_2$ has vertex set $V(G_1) \times V(G_2)$, and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G_1 \square G_2$ if and only if either $u_1 = u_2$ and $v_1 v_2 \in E(G_2)$ or $v_1 = v_2$ and $u_1 u_2 \in E(G_1)$. Let $G = G_1 \square G_2$. For a vertex $u \in V(G_1)$, denote by G_2^u the subgraph of G induced by all vertices (u, x) for $x \in V(G_2)$. Similarly, for a vertex $v \in V(G_2)$, the subgraph G_1^v of G is the graph induced by all vertices (x, v) for $x \in V(G_1)$.

The incidence coloring problem on the Cartesian product of some special classes of graphs has been extensively investigated, e.g., $P_m \square P_n$ [7,8], $P_m \square C_n$ [5,8], $C_m \square C_n$ [10], $P_m \square K_{n,h}$ [5], and Q_n [9], where P_m denotes a path of m vertices, C_n a cycle of n vertices, $K_{n,h}$ a complete bipartite graph with two vertex sets of n and h, respectively, vertices and Q_n the n-dimensional hypercube. In [11], Sun and Shiu showed that

$$\chi_i(G_1 \square G_2) \leqslant \chi_i(G_1) + \chi_i(G_2).$$

Definition 1.1. The *spectrum* of a vertex v with respect to incidence coloring σ, denoted by $S_\sigma(v)$, is the set consisting of all colors assigned to the incidences in $\mathrm{IN}(v)$. When the context is clear, we write $S(v)$ instead. Furthermore, let $s(\sigma) = \max\{|S_\sigma(v)| : v \in V(G)\}$.

In this paper, we show that if $n(\sigma_1) \geqslant n(\sigma_2) - s(\sigma_2)$, then there exists an incidence coloring of $G_1 \square G_2$ which uses $n(\sigma_1) + s(\sigma_2)$ colors; otherwise, there exists an incidence coloring of $G_1 \square G_2$ using $n(\sigma_2)$ colors, where σ_i is an incidence coloring of G_i for $i \in \{1, 2\}$. Based on the result above, we show that,

$$\chi_i(G_1 \square G_2) \leqslant \min\{\chi_i(G_1) - \delta(\sigma_2) + n(\sigma_2), \chi_i(G_2) - \delta(\sigma_1) + n(\sigma_1)\},$$

where σ_i is an incidence coloring of G_i with $s(\sigma_i)$ minimum among all incidence colorings of G_i for $i \in \{1, 2\}$, $\delta(\sigma_1) = \min\{\chi_i(G_2), n(\sigma_1) - s(\sigma_1)\}$, and $\delta(\sigma_2) = \min\{\chi_i(G_1), n(\sigma_2) - s(\sigma_2)\}$. Moreover, we also show that, for integer $p \geqslant 1$,

$$\chi_i(Q_n) = \begin{cases} n+1 & \text{if } n = 2^p - 1 \\ n+2 & \text{otherwise.} \end{cases}$$

2. Main results

Let σ_1 and σ_2 be incidence colorings of G_1 and G_2, respectively, and $G = G_1 \square G_2$. In Algorithm A, we describe how to find an incidence coloring of $G_1 \square G_2$ which uses $n(\sigma_1) + s(\sigma_2)$ colors if $n(\sigma_1) \geqslant n(\sigma_2) - s(\sigma_2)$; otherwise, $n(\sigma_2)$ colors will be used. For brevity, let $n_i = n(\sigma_i)$ and $s_i = s(\sigma_i)$ for $i \in \{1, 2\}$ and let $\eta_1 = \min\{n_2, n_1 - s_1\}$ and $\eta_2 = \min\{n_1, n_2 - s_2\}$. Furthermore, when $n_2 = \chi_i(G_2)$ (respectively, $n_1 = \chi_i(G_1)$), we use δ_1 (respectively, δ_2) to denote η_1 (respectively, η_2).

Example 1. We use $Q_{11} = Q_7 \square Q_4$ as an example to illustrate Algorithm A (see Fig. 1). In Step 1 of Algorithm A, by using the algorithm in [9], we have an incidence coloring σ_1 for Q_7. Note that, by using σ_1, the spectrum $S(v) = \{1, \ldots, 8\}$ for every vertex v in Q_7. There exists an incidence coloring σ_2 for Q_4 with $n_2 = 8$ and $s(\sigma_2) = 5$ (see Fig. 1(a)). This yields $\eta_2 = \min\{n_1, n_2 - s_2\} = \min\{8, 8 - 5\} = 3$. In Step 2 of Algorithm A, add $n_1 - \eta_2 = 8 - 3 = 5$ to every color assigned to the incidences of Q_4^u for $0 \leqslant u \leqslant 127$ (see Fig. 1(b)). In Step 3 of Algorithm A, the incidence colors in Q_7^v for $0 \leqslant v \leqslant 15$ have to be adjusted accordingly. For example, for every vertex $(u, 0)$ in Q_7^0 with $0 \leqslant u \leqslant 127$, there are adjacent incidences of $\mathrm{IN}((u, 0))$ using the same colors in $\{6, 7, 8\}$. Thus we have to adjust the colors assigned to the incidences in Q_7^0 with colors 6, 7, and 8 to avoid two adjacent incidences having the same color. Note that $\{n_1 + 1, \ldots, n_1 - \eta_2 + n_2\} = \{9, \ldots, 13\}$. We can find that colors 11, 12, and 13 are not used in $\mathrm{IN}((u, 0))$ for $0 \leqslant u \leqslant 127$. Thus we can use colors 11, 12, and 13 to replace colors 6, 7, and 8, respectively, in Q_7^0. Similarly, all incidences with colors 6, 7, and 8 in Q_7^1 are adjusted to 10, 12, and 13, respectively; all incidences with colors 6, 7, and 8 in Q_7^3 are adjusted to 10, 11, and 12, respectively; and so on. For every $S(u, v)$ in Fig. 1(c), the former set contains the colors used in Q_7^v and the latter contains the colors used in Q_4^u.

Algorithm A: An incidence coloring of Cartesian product graphs.

Input: Incidence colorings σ_1 and σ_2 of G_1 and G_2, respectively.
Output: An incidence coloring of $G_1 \square G_2$.
Step 1. /* Initialization. */
  For each $u \in V(G_1)$, assign a color to each incidence of G_2^u by σ_2.
  For each $v \in V(G_2)$, assign a color to each incidence of G_1^v by σ_1.
Step 2. /* Adjust the colors in G_2^u for each $u \in V(G_1)$. */
  Add $n_1 - \eta_2$ to the color assigned to each incidence in G_2^u for each $u \in V(G_1)$.
Step 3. /* Adjust the colors in G_1^v for each $v \in V(G_2)$. */
  For each vertex (u, v) in G, if there are two adjacent incidences of $\mathrm{IN}((u, v))$ having the same color, say t, then replace all incidences with color t in G_1^v by a color in $\{n_1 + 1, \ldots, n_1 - \eta_2 + n_2\}$ which is not used in $\mathrm{IN}((u, v))$.
Step 4. /* Output. */
  Let σ be the mapping of incidences and colors obtained in the previous steps.
  Output σ.

Now we are at a position to prove that the function σ obtained by Algorithm A is an incidence coloring of $G_1 \square G_2$ with $n(\sigma) = n_1 - \eta_2 + n_2$.

Lemma 2.1. *The function σ obtained by Algorithm A is an incidence coloring of $G_1 \square G_2$.*

Proof. We have to ensure that, in Step 3 of Algorithm A, there are enough colors in $\{n_1 + 1, \ldots, n_1 - \eta_2 + n_2\}$ which are not used in $\mathrm{IN}((u, v))$ so that we can replace all those adjacent incidences with the same color in G_1^v.

First we consider the case where η_2 is equal to n_1, namely $n_2 - s_2 > n_1$. If $s_2 \geqslant n_1$, then all n_1 colors used in the incidences of G_1^v have to be replaced (see Fig. 2(a)).