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# On Secure Domination in Graphs

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ABSTRACT

A set $D \subset V$ of a graph $G = (V, E)$ is a dominating set of $G$ if every vertex not in $D$ is adjacent to at least one vertex in $D$. A secure dominating set $S$ of a graph $G$ is a dominating set with the property that each vertex $u \in V \setminus S$ is adjacent to a vertex $v \in S$ such that $(S \setminus \{v\}) \cup \{u\}$ is a dominating set. The secure domination number $\gamma_s(G)$ equals the minimum cardinality of a secure dominating set of $G$. We first show that the problem of computing the secure domination number is NP-complete for bipartite and split graphs. Then we present bounds relating the secure domination number to the domination number $\gamma(G)$, the independence number $\beta_0(G)$ and the independent domination number $i(G)$. In particular, we prove that $\gamma_s(G) \le \gamma(G) + \beta_0(G) - 1$ if $G$ is an arbitrary graph, $\gamma_s(G) \le \frac{3}{2}\beta_0(G)$ if $G$ is triangle-free, and $\gamma_s(G) \le \beta_0(G)$ if $G$ has girth at least six. Finally, we show for the class of trees $T$ that $\gamma_s(T) \ge i(T)$ and $\gamma_s(T) > \beta_0(T)/2$. The last result answers the question posed by Mynhardt at the 22nd Clemson mini-Conference, 2007



## 1. Introduction

Let $G = (V, E)$ be a simple graph. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u \in V : uv \in E\}$ and the *closed neighborhood* of $v$ is the set $N[v] = \{v\} \cup N(v)$. For a set $X \subseteq V$, we denote by $G[X]$ the subgraph of $G$ induced by $X$. The open and closed neighborhoods of $X$ are $N(X) = \cup_{x \in X} N(x)$ and $N[X] = X \cup N(X)$, respectively. An $S$-*external private neighbor* of a vertex $v \in S$ is a vertex $u \in V \setminus S$ which is adjacent to $v$ but to no other vertex of $S$. The set of all $S$-external private neighbors of $v \in S$ is called the $S$-*external private neighbor set* of $v$ and is denoted $\text{epn}(v, S)$. The *degree* of $v$ is the cardinality of its open neighborhood. A vertex of degree one is called a *leaf* and its neighbor is called a *stem*.

A vertex in a graph $G$ is said *to dominate* itself and every vertex adjacent to it. A subset $D$ of $V$ is said to be a *dominating set* of $G$ if every vertex not in $D$ is adjacent to at least one vertex in $D$. The *domination number* $\gamma(G)$ equals the minimum cardinality of a dominating set in $G$. A set $S \subset V$ is *independent* if no two vertices in $S$ are adjacent. The minimum and maximum cardinalities, respectively, of a maximal independent set in $G$ equal the *independent domination number* $i(G)$ and the *independence number* $\beta_0(G)$, respectively.

A subset $S \subseteq V$ is a 2-*dominating set* if every vertex of $V \setminus S$ has at least two neighbors in $S$ and a *double dominating set* if $S$ is a 2-dominating set and the subgraph induced by $S$ has no isolated vertex. The 2-*domination number* $\gamma_2(G)$ and the *double domination number* $\gamma_{\times 2}(G)$ represent the cardinality of a minimum 2-dominating set and, respectively, of a minimum double dominating set of $G$. Clearly, $\gamma_{\times 2}(G) \ge \gamma_2(G)$ holds for every graph $G$ with no isolated vertices.

A *secure dominating set* (SDS) $S$ of a graph $G$ is a dominating set with the property that each vertex $u \in V \setminus S$ is adjacent to a vertex $v \in S$ such that $(S \setminus \{v\}) \cup \{u\}$ is a dominating set. The *secure domination number* $\gamma_s(G)$ equals

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the minimum cardinality of a secure dominating set of $G$. Secure domination was introduced by Cockayne et al. [5], and is studied, for example, in [5,7,9]. For any parameter $\mu(G)$ associated with a graph property $\mathcal{P}$, we refer to a set of vertices with Property $\mathcal{P}$ and cardinality $\mu(G)$ as a $\mu(G)$-set.

A *tree* is a connected graph that contains no cycle. We denote the *star* consisting of one central vertex and $k$ leaves as $K_{1,k}$. The *girth* of a graph $G$ is the length of a shortest cycle (if any) in $G$. If the graph does not contain any cycles, then its girth is defined to be infinity. If a graph $G$ does not contain an induced subgraph that is isomorphic to some graph $F$, then we say that $G$ is $F$-*free*. In particular, if $F = K_{1,3}$, we say that $G$ is *claw-free*, and if $F = K_3$, we say that $G$ is *triangle-free*. A *clique* of a graph $G$ is a complete subgraph of $G$. A graph is *split* if its vertex set can be partitioned into a clique and an independent set.

In this paper, we show that the problem of computing the secure domination number is in the NP-complete class, even when restricted to bipartite graphs and split graphs. Then we prove that for every graph $G$, $\gamma_s(G) \le \gamma(G) + \beta_0(G) - 1$, which improves a previous bound of Klostermeyer and Mynhardt [7]. Bounds relating the secure domination to the independence number are also presented for triangle-free graphs. More precisely, we show that if $G$ is a triangle-free graph, then $\gamma_s(G) \le \frac{3}{2}\beta_0(G)$ and if $G$ has girth at least six, then $\gamma_s(G) \le \beta_0(G)$. For the class of trees $T$, we show that $\gamma_s(T) \ge i(T)$ and $\gamma_s(T) > \beta_0(T)/2$.

Before presenting our results, we need to recall some results that are important for our investigations. We begin by giving a fundamental property of secure dominating sets due to Cockayne et al. [5].

**Proposition 1.1.** *(See Cockayne et al. [5].) If $D$ is a secure dominating set of a graph $G$, then for every vertex $v \in D$, the subgraph induced by* $\mathrm{epn}(v, D)$ *is complete.*

**Theorem 1.2.** *For every nontrivial tree $T$,*

i) *(See Blidia et al. [3].)* $\gamma_{\times 2}(T) \ge 2i(T)$.
ii) *(See Blidia et al. [2].)* $\gamma_2(T) \ge \beta_0(T)$, *with equality if and only if $T$ has a unique $\gamma_2(T)$-set that also is a $\beta_0(T)$-set.*

## 2. Complexity results

Our aim in this section is to establish the NP-complete results for the secure domination problem in split graphs and in bipartite graphs. For this purpose, we use a transformation from the domination problem.

DOMINATION PROBLEM (DOM).
**Instance:** Graph $G$ and a positive integer $k$.
**Question:** Does $G$ have a dominating set of cardinality at most $k$?

Garey and Johnson [6] were the first to show that the domination problem is NP-Complete for arbitrary graphs. Thereafter Bertossi [1] showed that the problem remains NP-Complete even when restricted to bipartite graphs or split graphs.

Now let us consider the following decision problem, to which we shall refer as SDOM.

SECURE DOMINATION PROBLEM (SDOM)
**Instance:** Graph $G^*$ and a positive integer $k^*$.
**Question:** Does $G^*$ have a secure dominating set of cardinality at most $k^*$?

**Theorem 2.1.** *Problem SDOM is NP-Complete for split graphs.*

**Proof.** Clearly, SDOM is a member of $\mathcal{NP}$, since we can check in polynomial time whether or not a set of vertices is a secure dominating set of $G$. Now let us show how a polynomial time algorithm for SDOM could be used to solve DOM in polynomial time. Given a positive integer $k$ and a split graph $G$ whose vertex set is partitioned into a clique $Q$ and an independent set $I$, we construct a split graph $G^*$ with clique $Q^*$ and independent set $I^*$ by adding a path $P_2 : x\text{-}y$ such that $x$ is adjacent to all vertices of $G$. Clearly $y$ is a leaf in $G^*$ and $x$ its stem. It is worth pointing out that $G^*$ is a split graph, where $V(Q^*) = V(Q) \cup \{x\}$ and $I^* = I \cup \{y\}$. We also note that $|V(G^*)| = |V(G)| + 2$ and $|E(G^*)| = |E(G)| + |V(G)| + 1$, and so $G^*$ can be constructed from $G$ in polynomial time.

Next, we shall show that $G$ has a dominating set $D$ with $|D| \le k$ if and only if $G^*$ has a secure dominating set $D^*$ with $|D^*| \le k^* = k + 1$. Let $D$ be a dominating set of size at most $k$. Then it is evident that $D^* = D \cup \{x\}$ is a secure dominating set of $G^*$ of cardinality $|D| + 1 \le k + 1$. Conversely, suppose that $G^*$ has a secure dominating set $D^*$ with $|D^*| \le k^* = k + 1$. Let $D' = D^* \cap V(G)$. It is clear that $|D^* \cap \{x, y\}| \ge 1$ and hence $|D'| \le k$. If $D'$ dominates $G$, then $D = D'$ and we are finished. Suppose now that $D'$ is not a dominating set of $G$. Let $A$ be the set of vertices of $G$ having no neighbor in $D'$ and let $w \in A$. Since $A$ is securely dominated by $D^*$ in $G^*$, we deduce that $x \in D^*$ and $A \subset \mathrm{epn}(x, D^*)$. Since $\mathrm{epn}(x, D^*)$ induces a complete graph, by Proposition 1.1, it follows that $y \in D^*$, and so $|D'| = |D^*| - 2$. Now it is clear that $D = D' \cup \{w\}$ is a dominating set of $G$ with $|D| = |D' \cup \{w\}| \le (k^* - 2) + 1 = k$. □

**Theorem 2.2.** *Problem SDOM is NP-Complete for bipartite graphs.*

**Proof.** Clearly, SDOM is a member of $\mathcal{NP}$. We next show how a polynomial time algorithm for SDOM could be used to solve DOM in polynomial time. Given a bipartite graph $G$ with partite sets $X$ and $Y$, and a positive integer $k$, we construct a bipartite graph $G^*$ with partite sets $X^*$ and $Y^*$ by adding two disjoint paths $P_2$, say $x_1\text{-}x_2$ and $y_1\text{-}y_2$ such that $x_1$ is adjacent to all $X$ and $y_1$ is adjacent to all $Y$. Note that $x_2$ and $y_2$ are leaves in $G^*$. Clearly $G^*$ is a bipartite graph, where $|V(G^*)| = |V(G)| + 4$ and $|E(G^*)| = |E(G)| + |V(G)| + 2$, and so $G^*$ can be constructed from $G$ in polynomial time.

We will show that $G$ has a dominating set $D$ with $|D| \le k$ if and only if $G^*$ has a secure dominating set $D^*$ with $|D^*| \le k^* = k + 2$. Let $D$ be a dominating set of $G$ of size at most $k$. Then clearly $D^* = D \cup \{x_1, y_1\}$ is a secure