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Vertex coloring edge-weighted digraphs ☆

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ABSTRACT

A coloring of a digraph with non-negative edge weights is a partition of the vertex set into independent sets, where a set is independent if the weighted in-degree of each node within the set is less than 1. We give constructive optimal bounds on the chromatic number in terms of maximum in-degree and inductiveness of the graph.

1. Introduction

Let $D = (V, E, w)$ be a digraph with an asymmetric weight function $w : E \rightarrow \mathbb{R}_0^+$ mapping edges to non-negative real numbers. Let $n = |V|$. The (weighted) *in-degree* of node v with respect to a set $S \subseteq V$ is $d_S^-(v) = \sum_{u \in S} w(u, v)$. A subset S of V is an *independent set* (or *color*) if $d_S^-(v) < 1$ holds for every v in S, i.e., if the in-degree of each node in S is strictly less than 1. A *coloring* of D is a partition of V into independent sets and the *chromatic number* $\chi(D)$ is the minimum number of colors needed on D. Observe that these definitions properly generalize independent sets and colorings in ordinary graphs, which correspond to the special case of 0-1 weight functions.

We explore here bounds on the chromatic number of edge-weighted digraphs in terms of degree parameters of the graph. We particularly focus on the maximum in-degree $\Delta^- = \Delta^-(D) = \max_{v \in V} d_V^-(v)$, but also consider the undirected measure of *inductiveness* $\tau(D) = \max_{H \subseteq D} \min_{v \in V(H)} d_H(v)$, where $d_H(v) = \sum_{u \in V(H)} (w(u, v) + w(v, u))$.

Previous work This problem has origin in the scheduling of wireless communication links under the SINR model of interference. Tamura et al. [17] appear to have been the first to propose this edge-weighted graph formulation, although recent work has drawn on the rediscovery of Hoefer, Kesselheim and Vöcking [13]. Each node in a conflict graph corresponds to a communication *link* and the weight of the edge from u to v corresponds to the relative interference (or *affectance* [11]) of link u on link v. A set of links is feasible if all the links can successfully communicate simultaneously. The feasibility of a link set corresponds precisely to independence in the link conflict graph.[3]

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[3] A minor caveat is that in scheduling studies, feasibility corresponds to a set where the weighted in-degree of each link is at most 1 (not necessarily strictly less).

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The link scheduling problem is usually studied in a metric space. This naturally constrains the possible edge weightings, which significantly impacts the computational tractability of the problem. In fact, our independent set problem is constant-factor approximable in a metric setting [10], whereas in arbitrary ordinary graphs the problem is hard to approximate within $\Omega(n^{1-\epsilon})$ factor, for any $\epsilon > 0$ [12].

The link scheduling problem was first posed as an algorithmic problem by Moscibroda and Wattenhofer in 2006 [16]. It was shown to be NP-complete by Goussevskaia et al. [8], even to determine if $\chi(D) \le 2$ (for a special subclass of metric instances). While most results known apply only to specific metric settings, there are some results known for the general case. A bound of $\chi(D) \le \lfloor 2\Delta^- + 1 \rfloor^2$ was given in [11], attained by a simple sequential algorithm. When Δ^- is sufficiently large, a randomized distributed algorithm attains an asymptotically stronger bound: $\chi(D) = O(\Delta^- + \log^2 n)$ [7].

Related problems have been studied in the graph literature. A *d-defective* coloring (of an unweighted graph) [5] is one where each node can have at most d neighbors in the same color class (also known as *d-improper* coloring). A similar problem was also recently considered by Araujo et al. [2]. Numerous papers have been written on this problem, especially regarding families of planar graphs. It can be viewed as a restriction of our problem to the case when all the discretized weights are equal and same in both directions. An old (and frequently rediscovered) algorithm of Lovász [15] shows that for this symmetric case, $\Delta^- = \Delta^+$ is an upper bound on the number of colors needed.

The asymmetric edge-weighted coloring problem we address (sometimes assumed to be in a discretized form) was first treated by Tamura et al. [17], who derived some basic properties, such as how the chromatic number distributes over connected and biconnected components. Recently, Archetti et al. [3] gave a branch-and-bound algorithm.

Our contributions We give constructive bounds on the edge-weighted chromatic number in terms of the degree parameters of the graph: maximum in- and out-degree, and inductiveness. The bounds are essentially tight. The results have implications for the theory of wireless scheduling in the SINR model.

In Section 2, we build on a result of Alon to obtain an upper bound in terms of maximum in-degree. We then show in Section 3 that stronger lower bounds hold in terms of the other degree parameters, whereas a better bound can be obtained for the corresponding independence number of sparse instances. The applications to SINR theory are indicated in Section 4 before closing off with conclusions.

2. Bounds in terms of maximum in-degree

We obtain an essentially tight bound on $\chi(D)$ in terms of the maximum in-degree. We need the following lemma of Alon [1] that generalizes a result that he attributes Keith Ball citing [4].

Lemma 1. *Let $A = (a_{ij})$ be an n by n real matrix, where $a_{ii} = 0$ for all i, $a_{ij} \ge 0$ for all $i \ne j$, and $\sum_j a_{ij} \le 1$ for all i. Then, for every k and positive reals $c_1, \ldots, c_k$ whose sum is 1, there is a partition of $[n] = \{1, 2, \ldots, n\}$ into pairwise disjoint sets $S_1, S_2, \ldots, S_k$, such that for every r, $1 \le r \le k$ and every $i \in S_r$, we have $\sum_{j \in S_r} a_{ij} \le 2c_r$.*

Using this lemma, we get the following.

Theorem 1. *For every digraph D, $\chi(D) \le \lfloor 2\Delta^- + 1 \rfloor$.*

Proof. Given D with $|V(D)| = n$, form the matrix (a_{ij}) where $a_{ij} = w(v_j, v_i)/\Delta^-$. Let $k = \lfloor 2\Delta^- + 1 \rfloor$ and define $c_r = 1/k$, for $1 \le r \le k$. These parameters satisfy the conditions of Lemma 1. Let $S_1, S_2, \ldots, S_k$ be the partition of $V(D)$ resulting from applying Lemma 1 with these parameters. It then holds for each $1 \le r \le k$ and each $v_i \in S_r$ that

$$d^-_{S_r}(v_i) = \sum_{v_j \in S_r} w(v_j, v_i) = \Delta^- \sum_{v_j \in S_r} a_{ij} \le \Delta^- \frac{2}{k}$$
$$= \frac{2\Delta^-}{\lfloor 2\Delta^- + 1 \rfloor} < 1 \,.$$

Hence, the partition is a valid coloring. □

This turns out to be a tight bound.

Proposition 1. *For every natural number t, there is a digraph D with $\Delta^-(D) = t$ and $\chi(D) = 2\Delta^- + 1$.*

Proof. Consider a regular tournament T_n with $n = 2k + 1$, i.e., where each vertex has in- and out-degree k. Then, viewing the edges as having weight 1, we see that each node must receive a different color. □

With a slight increase in the number of colors, we can obtain an algorithmic version.

Lemma 2. *Let $q > 0$ and let $A = (a_{ij})$ be an n by n real matrix, where $a_{ii} = 0$ for all i, $a_{ij} > q$ for all $i \ne j$, and $\sum_j a_{ij} \le 1$ for all i. Also let k be a number and $\epsilon > 0$. There is an algorithm running in time polynomial in n, $1/q$, and $1/\epsilon$ that finds a partition of $[n] = \{1, 2, \ldots, n\}$ into disjoint sets $S_1, S_2, \ldots, S_k$, such that for every r, $1 \le r \le k$ and every $i \in S_r$, we have $\sum_{j \in S_r} a_{ij} \le 2/k + \epsilon/q$.*

Proof. We follow closely Alon's proof of Lemma 1. By increasing some of the numbers a_{ij}, if needed, we may assume that $\sum_j a_{ij} = 1$ for all i. Thus, by the Perron–Frobenius Theorem, 1 is the largest eigenvalue of A, with right eigenvector $(1, 1, \ldots, 1)$, and A has a left eigenvector $(u_1, u_2, \ldots, u_n)$ in which all entries are positive and $\sum_j u_j = 1$. It follows that for all j, $\sum_i u_i a_{ij} = u_j$. Observe that for all j, $u_j = \sum_i u_i a_{ij} \ge q \sum_i u_i = q$.

Define $b_{ij} = u_i a_{ij}$, and note that $\sum_i b_{ij} = u_j$ and $\sum_j b_{ij} = u_i(\sum_j a_{ij}) = u_i$. Define the potential function Φ that, given a partition $\Pi = (S_1, S_2, \ldots, S_k)$ of $[n]$ into k disjoint sets, has value