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Mutual witness proximity graphs

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ABSTRACT

This paper describes one variation on witness proximity graphs called mutual witness proximity graphs. Two witness proximity graphs are said to be mutual when, given two sets of points A and B, A is the vertex set of the first graph and the witness set of the second one, while B is the witness set of the first graph and the vertex set of the second one. We show that in the union of two mutual witness Delaunay graphs, there are always at least $\lceil \frac{n-2}{2} \rceil$ edges, where $n = |A| + |B|$, which is tight in the worst case. We also show that if two mutual witness Delaunay graphs are complete, then the sets A and B are circularly separable; if two mutual witness Gabriel graphs are complete, then the sets A and B are linearly separable; but two mutual witness rectangle graphs might be complete, with A and B not linearly separable.

1. Introduction

A *proximity graph* is a graph whose vertices correspond to some geometric objects, and an edge connects two of them when the objects are considered to be *neighbors* according to some proximity criterion. These graphs have been widely used for spatial data analysis [1,2], pattern recognition (see [3,4] and references therein), data mining [5], and in many disciplines where classification or interpolation is required [2,4,6]. On the other hand, as connecting neighbors naturally yields an aesthetically appealing layout of a graph, proximity graphs have also been a topic thoroughly studied in the area of graph drawing [7,8].

We consider proximity graphs on point sets on the plane. Given a point set S, the Delaunay graph $\mathrm{DG}(S)$, the Gabriel graph $\mathrm{GG}(S)$, and the rectangle of influence graph $\mathrm{RIG}(S)$ of S all have vertex set S. Given $p, q \in S$, there is an edge connecting p and q in $\mathrm{DG}(S)$ if there is some disk with p and q on its boundary, whose interior is empty of points from S. The vertices p and q are adjacent in $\mathrm{GG}(S)$ if the closed disk with diameter pq covers no other points from S. Finally, there is an edge connecting p and q in $\mathrm{RIG}(S)$ if the closed axis-aligned rectangle $B(p, q)$ with p and q as opposite corners contains no other points from S. These graphs belong to the *Delaunay family* of graphs, and define neighbor relationships that describe how the points interact; see [3] for a survey on the topic and its applications.

The rectangle of influence graph was introduced by Ichino and Sklansky in [9]. In the same work they also defined the *mutual neighborhood graph of a point set A against a second point set B*, denoted $\mathrm{MNG}(A|B)$: its vertex set is A, and there is an edge connecting $p, q \in A$ when the rectangle $B(p, q)$ contains no points from B (with the exception of p and q themselves, should they belong to B). The motivation behind their definition was to consider the union of the graphs $\mathrm{MNG}(A|B)$ and $\mathrm{MNG}(B|A)$ as a descriptor of the interaction between the sets A and B.

A systematic study of *witness proximity graphs*, which are proximity graphs on a point set with respect to a second point set, was undertaken in [10–13] and also studied later in other works such as [14] and [15]. For the sake of

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conciseness we don't give here the general definition but concentrate on the two graphs we study in the following sections.

We define the *witness Delaunay graph* of a point set A of *vertices* in the plane, with respect to a point set B of *witnesses*, denoted $DG^-(A,B)$, as the graph with vertex set A in which two points $x, y \in A$ are adjacent when there is an open disk whose boundary passes through x and y and that does not contain any witness $w \in B$. Notice that we don't care about the presence of elements from $A \setminus B$ inside the disk. $DG^-(A,B)$ is a *negative-witness graph* because the witnesses from B "prevent" adjacencies. $DG^-(A,\varnothing)$ is simply the complete graph $K_{|A|}$. $DG^-(A,A)$ is precisely the Delaunay graph $DG(A)$, which under standard non-degeneracy assumptions is a triangulation, denoted by $DT(A)$.

For a pair of points p, q in the plane, we denote by D_{pq} the closed disk with diameter pq. The *witness Gabriel graph* $GG^-(A,B)$ is defined by two sets of points A and B; A is the set of vertices of the graph and B is the set of *witnesses*. There is an edge xy in $GG^-(A,B)$ if, and only if, there is no point of B in $D_{xy} \setminus \{x,y\}$. It is a negative witness graph. $GG^-(A,A)$ is precisely the original Gabriel graph $GG(A)$.

Following the rationale of Ichino and Sklansky [9], it is natural to wonder what would be the characteristics of the two interdependent witness graphs that we call *mutual witness graphs*, corresponding to the witness Delaunay graphs and to the witness Gabriel graphs. In both cases the two graphs would be mutual in the following sense: Given two sets of points A and B, A is the vertex set of the first graph and the witness set of the second one, while B is the witness set of the first graph and the vertex set of the second one.

In Sections 2 and 3 we consider the properties of the mutual witness Delaunay graphs and the mutual witness Gabriel graphs, respectively, the first one being the main subject of this work. We conclude in Section 4 with an observation on the mutual neighborhood graphs originally introduced by Ichino and Sklansky. We assume hereafter *strong general position* of the point sets A and B: no three collinear points and no four concyclic points. Removing this assumption involves complicating some arguments and invalidating other theorems below; for several of the objects there are no agreed upon standard definitions in presence of degeneracies.

2. Mutual witness Delaunay graphs

We first consider the joint size of the mutual witness Delaunay graphs $DG^-(A,B)$ and $DG^-(B,A)$:

Theorem 1. *Given disjoint sets of points A and B, the number of edges in $DG^-(A,B) \cup DG^-(B,A)$ is at least $\lceil \frac{n-2}{2} \rceil$, where $n = |A| + |B|$. This bound is tight for $n \geq 2$.*

Proof. For the lower bound, color points of A black and points of B white. Consider $DT(A \cup B)$, the Delaunay triangulation of $A \cup B$. Each triangle has at least one *monochromatic edge*, i.e., an edge incident to two vertices of the same color. A monochromatic edge pq belongs to $DG^-(A,B) \cup DG^-(B,A)$ as there exists a disk empty of points of $A \cup B$ whose boundary circle passes through p and q. Therefore, the number of monochromatic edges in $DT(A \cup B)$ is a lower bound on the number of edges in $DG^-(A,B) \cup DG^-(B,A)$.

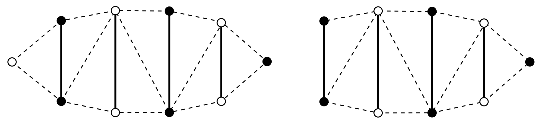

Fig. 1. The union of the witness Delaunay graphs $DG^-(A,B)$ and $DG^-(B,A)$ in each of the figures has $\lceil \frac{n-2}{2} \rceil$ monochromatic edges and $\lceil \frac{n+1}{2} \rceil$ components. Monochromatic edges of $DT(A \cup B)$ are drawn solid and bichromatic ones dashed.

Any Delaunay triangulation has at least $n-2$ triangles, each with at least one monochromatic edge. Since a monochromatic edge might appear in two faces, and if the number of faces is odd at least one such edge is not counted twice, the number of monochromatic edges is at least $\lceil \frac{n-2}{2} \rceil$.

To see that this bound is tight, we present a concrete construction; refer to Fig. 1. Place ℓ black points and m white points, with $\ell + m = n$ and $\ell = m$ ($\ell = m-1$ in the odd case), spaced on the boundary of an ellipse in an alternating manner and such that, except for the leftmost point (for the even case only) and rightmost point, pairs of black points and pairs of white points are placed at the same x-coordinate.

Consider $DT(A \cup B)$. The vertical edges between pair of points at the same x-coordinate define the only monochromatic edges in $DT(A \cup B)$. Indeed, for any pair of points a and b of the same color on different sides of a vertical monochromatic edge, the edge ab would cross a monochromatic edge de of $DT(A \cup B)$. As de is in $DT(A \cup B)$, there exists a disk with d and e on its boundary that does not contain a or b. Hence, any disk with a and b on its boundary would contain either d or e or both. Therefore, there is no non-vertical monochromatic edge ab in $DG^-(A,B)$.

Thus the only edges in $DG^-(A,B) \cup DG^-(B,A)$ are the vertical ones connecting same-color vertices; and hence, the number of edges of $DG^-(A,B) \cup DG^-(B,A)$ is exactly $\lceil \frac{n-2}{2} \rceil$. □

Our next result, on a particular kind of geometric separability, proves that mutual witness Delaunay graphs contain useful information about interclass structure.

Theorem 2. *For disjoint sets of points A and B, if $DG^-(A,B)$ and $DG^-(B,A)$ are complete, then the sets A, B are circularly separable. The converse is not true.*

Proof. Consider the convex hulls $CH(A)$ and $CH(B)$. If they don't intersect, A and B are linearly separable and therefore circularly separable.

Suppose $CH(A)$ and $CH(B)$ intersect and an edge ab of $DG^-(A,B)$ crosses an edge cd of $DG^-(B,A)$. If the points a, c, b, d are in this clockwise order, consider $CH(\{a,c,b,d\})$: either $\measuredangle acb + \measuredangle bda > 180°$, or $\measuredangle cbd + \measuredangle dac > 180°$; hence, all disks D_{ab} with a, b on their