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Fast algorithms for some dominating induced matching problems

Min Chih Lin [a], Michel J. Mizrahi [a,*], Jayme L. Szwarcfiter [b,c]

[a] Departamento de Computación, Universidad de Buenos Aires, Argentina
[b] Instituto de Matemática, NCE and COPPE, Universidade Federal do Rio de Janeiro, Brazil
[c] Instituto Nacional de Metrologia, Qualidade e Tecnologia, Brazil

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ABSTRACT

We describe $O(n)$ time algorithms for finding the minimum weighted dominating induced matching of chordal, dually chordal, biconvex, and claw-free graphs. For the first three classes, we prove tight $O(n)$ bounds on the maximum number of edges that a graph having a dominating induced matching may contain. By applying these bounds, and employing existing $O(n+m)$ time algorithms we show that they can be reduced to $O(n)$ time. For claw-free graphs, we describe a variation of the existing algorithm for solving the unweighted version of the problem, which decreases its complexity from $O(n^2)$ to $O(n)$, while additionally solving the weighted version. The same algorithm can be easily modified to count the number of DIM's of the given graph.

1. Introduction

We consider undirected simple graphs G, denoting by $V(G)$ and $E(G)$, respectively, the sets of vertices and edges of G, $n = |V(G)|$ and $m = |E(G)|$. For $v \in V(G)$, $N(v)$ represents the set of neighbors of $v \in V(G)$, while $N[v] = N(v) \cup \{v\}$. For $S \subseteq V(G)$, $N(S) = \cup_{v \in S} N(v)$. We say a vertex $v \in V(G)$ such that $N[v] = V(G)$ is *universal*. Denote by $G[S]$ the subgraph of G induced by the vertices of S. If $G[S]$ is a 0-regular graph then S is an *independent set*, if it is a 1-regular graph then S is the set of vertices of an *edge independent set*. By $G + H$ we denote the disjoint union of two graphs G and H. We say that a graph G is H-free if G does not contain H as an induced subgraph. A vertex v is called *simplicial* if all its neighbors are adjacent to each other. An edge independent set is also known as an *induced matching*. For convenience, we may write induced matching to refer either to the set of edges or to its corresponding vertex set. Finally, we also employ the notation *matching* with its usual meaning of a set of pairwise non-adjacent edges.

Say that an edge $e \in E(G)$ *dominates* itself and every other edge adjacent to it. An *edge dominating set* of G is a set of edges $E' \subseteq E(G)$, such that every $e \in E(G)$ is dominated by some edge of E'. If each $e \in E(G)$ is dominated by exactly one edge of E' then E' is an *efficient edge dominating set*. In the latter situation, E' defines an induced matching, while the set of vertices not incident to E' form an independent set. For this reason, an efficient edge dominating set is also called *dominating induced matching (DIM)*. Not every graph admits a DIM. The *DIM problem* is to determine whether a graph has such a matching, and is known to be NP-complete [9]. We will consider graphs G with a weighting Ω, that assigns to each edge $vw \in E(G)$ a non-negative finite weight $\omega(vw)$. The aim is to find the minimum weight of a dominating induced matching of G, if any. We name this problem as $DIM_{\Omega}(G)$. Some of the existing algorithms for solving DIM problems are [3–5,10].

* Corresponding author.
E-mail addresses: oscarlin@dc.uba.ar (M.C. Lin), michel.mizrahi@gmail.com (M.J. Mizrahi), jayme@nce.ufrj.br (J.L. Szwarcfiter).

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Since the number of edges of any DIM of G, if existing, is invariant, it is straightforward to generalize the problem for edges with negative weights too.

Following the definition, the DIM problem can be viewed as to decide whether there is a partition of the vertices into two sets (say a *coloring* of the vertices in *white* and *black*) such that the white set is an independent set while the black one induces a 1-regular graph. Moreover, the black set defines a DIM of the graph [7]. A coloring is *partial* if only part of the vertices of G have been assigned colors, otherwise it is *total*. A black vertex is *single* if it has no black neighbor, and is paired if it has exactly one black neighbor. Each coloring, partial or total, can be *valid* or *invalid*.

A partial coloring is valid whenever any two white vertices are non-adjacent and each black vertex is either paired, or is single having some uncolored neighbor. A total coloring is valid whenever any two white vertices are non-adjacent and each black vertex is paired.

A valid partial coloring Γ might possibly extend into a coloring $\Gamma' \supseteq \Gamma$ by iteratively applying a set of coloring rules, compatible with Γ. In general, such rules would color some uncolored vertex v, whose color is uniquely determined by the colors of Γ. For instance, any uncolored neighbor of a white vertex must be colored as black, otherwise the coloring would be invalid. See [7] for a set of such rules. We refer to this process as *propagation*.

We prove that any chordal graph containing a DIM has at most $2n-3$ edges. Counting the edges and applying the $O(n+m)$ time algorithm by Lu, Ko and Tang [11] lead to an $O(n)$ time algorithm. For dually chordal graphs, by employing the similarity result *chordal - dually chordal* for DIM's by Brandstädt, Leitert and Rautenbach [2] also leads to solving the DIM problem in $O(n)$ time. For biconvex graphs, we prove that any $K_{3,3}$-free convex graph contains at most $2n-4$ edges. Additionally, that any biconvex graph containing a DIM is $K_{3,3}$-free. Using these two results, counting the number of edges of the given graph and employing the $O(n+m)$ time algorithm by Brandstädt, Hundt and Nevries [1] leads to solving the DIM problem for biconvex graphs in $O(n)$ time. Finally, for claw-free graphs, we describe a variation of the algorithm by Cardoso, Korpelainen and Lozin [7]. The latter solves the DIM problem, without weights, in $O(n^2)$ time, while the presently proposed algorithm requires $O(n)$ time for solving $DIM_{\Omega}(G)$.

A conference version of this paper has been presented at *LATIN' 2014* [6].

2. Chordal, dually chordal and biconvex graphs

In this section, we remark that computing $DIM_{\Omega}(G)$ for any graph G which is chordal, dually chordal or biconvex requires no more than $O(n)$ time.

Lemma 1. *(See [1].) If G contains a K_4 then G has no DIM's.*

Lemma 2. *Every K_4-free chordal graph G with at least 2 vertices has at most $2n-3$ edges. The bound is tight even if G is an interval graph.*

Proof. By induction on the number of vertices. For $n=2$, the result is trivial. Suppose the bound is valid for graphs with $n-1$ vertices, $n \geq 3$. Let G be an n-vertex chordal graph and v a simplicial vertex of it. Since $|E(G)| = |E(G \setminus \{v\})| + d(v)$, where $d(v)$ denotes the degree of v, by the induction hypothesis, the number of edges of $G \setminus \{v\}$ is bounded by $2n-5$. Since G is K_4-free, $d(v) \leq 2$, therefore $|E(G)| \leq 2n-5+2 = 2n-3$.

An interval graph having two universal vertices and the remaining ones having degree 2 has no K_4 and contains $2n-3$ edges, meaning that the bound is tight for interval graphs. □

Corollary 3. *The $DIM_{\Omega}(G)$ problem can be solved in $O(n)$ time for (dually) chordal graphs.*

Proof. Let G be a given chordal graph. First, count the number of edges of G, up to a limit of $2n-3$. If the bound has been exceeded then stop answering that G has no DIM's. Otherwise, apply the algorithm [11] which solves $DIM_{\Omega}(G)$ in $O(n)$ time. Finally, if a graph has a DIM then it is chordal if and only if it is dually chordal [2]. Consequently, $DIM_{\Omega}(G)$ can also be solved in $O(n)$ time for dually chordal graphs. □

Next, consider solving $DIM_{\Omega}(G)$ for biconvex graphs.

An ordering $<$ of X in a bipartite graph $G=(X,Y,E)$ has the *interval property* if for every vertex $y \in Y$, the vertices of $N(y)$ are consecutive in the ordering $<$ of X. A bipartite graph (X,Y,E) is *convex* if there is an ordering of X or Y that fulfills the interval property. Furthermore if there are orderings for both X and Y which fulfill the interval property the graph is *biconvex*.

Lemma 4. *Let G be a convex bipartite graph having no subgraph isomorphic to $K_{3,3}$. Then G contains at most $2n-4$ edges, for $n \geq 3$.*

Proof. By induction on n. If $n=3$, the graph has at most 2 edges, satisfying the bound. Let G be an arbitrary $K_{3,3}$-free convex graph, v its minimum degree vertex and G' the graph obtained from G by removing v.

- $d(v) \leq 2$: Clearly, G' is also $K_{3,3}$-free. By inductive hypothesis, G' has at most $2n-6$ edges. Consequently, G has at most $2n-6+d(v) \leq 2n-4$ edges.
- $d(v) > 2$: Every vertex in G has degree at least 3. Let $G=(X,Y,E)$ where X has the interval property. Thus for each vertex $y \in Y, N(y)$ consists of vertices that are consecutive. Let $\{x_1,\dots,x_k\}$ be the ordering $<$ of X and w.l.o.g. let $\{y_1,y_2,y_3\} \subseteq N(x_1)$. Since y_1, y_2, y_3 have at least 3 neighbors and X has the interval property, it follows that $\{x_2,x_3\} \subseteq N(y_1) \cap N(y_2) \cap N(y_3)$. Therefore $\{x_1,x_2,x_3,y_1,y_2,y_3\}$ induces a $K_{3,3}$, a contradiction.

Hence, G contains indeed at most $2n-4$ edges. This bound is tight, $K_{2,n-2}$ is an example. □

We remark that bipartite graphs, not necessarily convex, which do not contain $K_{3,3}$ as a minor also have at