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Counting maximal independent sets in directed path graphs

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ABSTRACT

The problem of counting maximal independent sets is #P-complete for chordal graphs but solvable in polynomial time for its subclass of interval graphs. This work improves upon both of these results by showing that the problem remains #P-complete when restricted to directed path graphs but that a further restriction to rooted directed path graphs admits a polynomial time solution.

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1. Introduction

The problem of counting maximal independent sets (abbr. #MIS problem) for general graphs is well-known to be #P-complete [1]. Valiant [2] defined the class of #P problems as those that involve counting access computations for problems in *NP*, while the class of #P-complete problems includes the hardest problems in #P. As is well known, all algorithms for exactly solving these problems have exponential time complexity, so efficient algorithms for solving this class of problems are unlikely to be developed. However, this complexity can be reduced by considering only a restricted subclass of #P-complete problems.

Some definitions and notation associated with graph theory are introduced as follows. For a graph G, let $V(G)$ and $E(G)$ denote its vertex and edge sets respectively. For a subset X of $V(G)$, $G[X]$ is the subgraph that is induced by the vertices of X and $G - X$ is the subgraph $G[V(G)\backslash X]$. An *independent set* (abbr. IS) in a graph G is a subset S of $V(G)$ such that no two vertices in S are adjacent. A *maximal independent set* (abbr. MIS) of a graph is an IS that is not a subset of any other IS in the graph. A *vertex cover* in a graph G is a subset C of $V(G)$ such that every edge in $E(G)$ has at least one endpoint in C. Clearly, C is a vertex cover of G if and only if $V(G) - C$ is an IS of G. A vertex u is said to *dominate* a vertex v if (u, v) is an edge in G. A *dominating set* in a graph G is a subset D of $V(G)$ such that every vertex that is not in D is dominated by at least one vertex in D. An *independent dominating set* of a graph G is a subset of vertices of G that is both independent and dominating in G. Obviously, a subset of vertices of a graph G is an independent dominating set if and only if it is a MIS of G. A *clique* in a graph G is a subset K of $V(G)$ such that each pair of vertices in K is connected by an edge. A *maximal clique* of a graph is a clique that is not a subset of any other clique in the graph.

Let F be a finite family of non-empty sets. A graph G is an *intersection graph* for F if a one-to-one correspondence exists between the vertices of G and the sets of F such that two vertices are adjacent if and only if their corresponding sets of F have a non-empty intersection. The class of intersection graphs has various important subclasses. Some of them are briefly described below.

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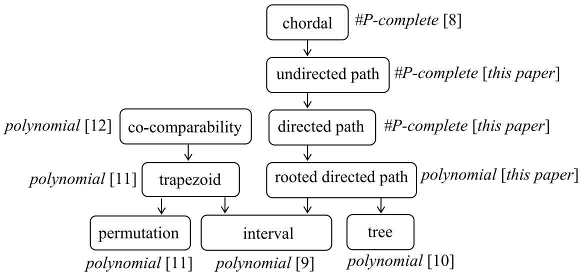

Fig. 1. Current status of problems of #MIS for some intersection graphs.

Chordal graphs are graphs in which every cycle with a length of at least four has a chord. Gavril [3] proved that chordal graphs are the intersection graphs of a family of subtrees in a clique tree. A tree T is a *clique tree* for a graph G if each node in T corresponds to a maximal clique in G and such that, for each vertex v of G, those nodes of T that contain v induce a subtree of T. For $v \in V(G)$, let T_v be the set of all maximal cliques of G that contain vertex v. Therefore, G is a chordal graph if and only if T_v is a subtree in a clique tree T for every $v \in V(G)$ [3]. In this way, four subclasses of chordal graphs can be defined [4]. *Undirected path graphs* (abbr. UPGs) are the intersection graphs of a family of undirected subpaths in a clique tree. *Directed path graphs* (abbr. DPGs) are the intersection graphs of a family of directed subpaths in a directed clique tree. *Rooted directed path graphs* (abbr. RDPGs) are the intersection graphs of a family of directed subpaths in a rooted directed clique tree. A tree is called a rooted directed tree if one node has been designated as the root, and the edges have a natural orientation, away from the root. *Interval graphs* are RDPGs in which the clique tree is itself a path. Interval graphs are typically defined as the intersection graphs of a family of intervals on the real line. It should be noted that every tree is a RDPG.

A *permutation graph* has an intersection model that consists of straight lines (one per vertex) between two parallel lines. *Trapezoid graphs* are the intersection graphs of a family of trapezoids (one per vertex) between two parallel lines [5]. If every trapezoid is a line, then the intersection graph is a permutation graph. Similarly if every trapezoid is a rectangle, then the intersection graph is an interval graph. Thus, trapezoid graphs properly include both interval and permutation graphs. A *co-comparability graph* is the complement of a comparability graph [6]. Co-comparability graphs are the intersection graphs of a family of curves (one per vertex) between two parallel lines [7].

The #MIS problem remains #P-complete even for chordal graphs [8], but an $O(n)$ time algorithm exists for interval graphs [9] and trees [10]; an $O(n^2)$ time algorithm exists for trapezoid and permutation graphs [11], and an $O(n^{2.3727})$ time algorithm exists for co-comparability graphs [12] (where n is the number of vertices). The status of the #MIS problem for UPGs, DPGs, and RDPGs has been open until now. This paper reveals that the #MIS problem remains #P-complete even when restricted to DPGs but that a stricter restriction to RDPGs admits an $O(n^3)$ time solution. Fig. 1 summarizes the situation. Consequently, the borderline between polynomial and #P-complete problems is fully determined for the graph classes in Fig. 1.

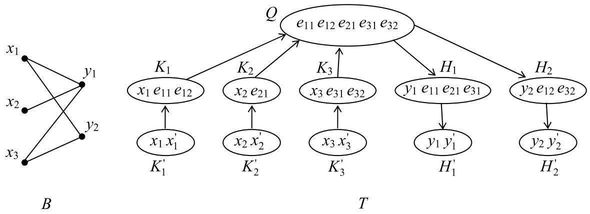

Fig. 2. A bipartite graph B and constructed clique tree T.

2. #P-completeness of the #MIS problem for directed path graphs

The #MIS problem for chordal graphs has been shown to be #P-complete by reduction from the problem of counting set covers [8]. A slight variation of that reduction suffices to prove that, even when restricted to DPGs, the #MIS problem remains #P-complete.

Theorem 1. *The #MIS problem for DPGs is #P-complete.*

Proof. The reduction is performed from the problem of counting vertex covers in a bipartite graph, which was proved to be #P-complete by Provan and Ball [13]. Let B be an arbitrary bipartite graph with bipartition $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_m\}$. In the following steps, the corresponding clique tree T is constructed from B, from which a DPG G is obtained. First, corresponding to each vertex $x_i \in X$, construct two cliques $K_i = \{x_i\} \cup \{e_{ij} | (x_i, y_j) \in E(B)\}$ and $K_i' = \{x_i, x_i'\}$. Next, corresponding to each vertex $y_j \in Y$, construct two cliques $H_j = \{y_j\} \cup \{e_{ij} | (x_i, y_j) \in E(B)\}$ and $H_j' = \{y_j, y_j'\}$. Finally, construct one large clique $Q = \{e_{ij} | (x_i, y_j) \in E(B)\}$. Let T be the directed clique tree such that $V(T)$ is the vertex set of all maximal cliques of the above construction and $E(T) = \{\langle K_i, Q\rangle | 1 \le i \le n\} \cup \{\langle Q, H_j\rangle | 1 \le j \le m\} \cup \{\langle K_i', K_i\rangle | 1 \le i \le n\} \cup \{\langle H_j, H_j'\rangle | 1 \le j \le m\}$ are directed edges of T. Fig. 2 presents an example of the above construction. Let G be the graph whose maximal cliques are the set $V(T)$. The resulting graph G, corresponding to the clique tree T, will now be shown to be a DPG. Obviously, $V(G) = X \cup Y \cup \{x_1', x_2', \ldots, x_n'\} \cup \{y_1', y_2', \ldots, y_m'\} \cup \{e_{ij} | (x_i, y_j) \in E(B)\}$. Let T_v be the set of all maximal cliques of G that contain vertex $v \in V(G)$. If $v = x_i(y_j)$ then T_v consists of the directed path $\langle K_i', K_i\rangle(\langle H_j, H_j'\rangle)$ of length one in T. If $v = x_i'(y_j')$, then T_v consists of the single vertex $K_i'(H_j')$ and it is a directed path of length zero in T. If $v = e_{ij}$, then T_v consists of the directed path $\langle K_i, Q, H_j\rangle$ of length two in T. Accordingly, for each vertex $v \in V(G)$, T_v is a directed path in T. Therefore, G is a DPG.

Now, the correspondence between the number of MISs in G and the number of vertex cover in B is established. Let M be a MIS in G that contains a vertex e_{ij} in clique Q.

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