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# Minimal $k$-rankings and the rank number of $P_n^2$

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## 1. Introduction

A $k$-ranking of a graph is a vertex labeling using integers between 1 and $k$ inclusive such that any path between two vertices of the same rank contains a vertex of strictly larger rank. A ranking $f$ has a *drop vertex* $x$ if the labeling defined by $g(v) = f(v)$ when $v \neq x$ and $g(x) < f(x)$ is still a ranking. It was shown by Jamison [9] that a ranking is minimal if and only if it contains no drop vertices. The *rank number* of a graph $G$ is the smallest $k$ such that $G$ has a minimal $k$-ranking. The *arank number* of a graph $G$ is the largest $k$ such that $G$ has a minimal $k$-ranking. When the value of $k$ is unimportant, we will refer to a $k$-ranking simply as a ranking.

Recall that a vertex coloring of a graph is a vertex labeling in which no two adjacent vertices have the same label. Hence a $k$-ranking is a vertex coloring with an additional condition imposed. Then similar to the chromatic number, the *rank number* $\chi_r(G)$ is the smallest $k$ such that $G$ has a minimal $k$-ranking. The *arank number* $\psi_r(G)$ is the largest $k$ such that $G$ has a minimal $k$-ranking.

The study of the rank number was motivated by applications including the design of very large scale integration (VLSI) layout and Cholesky factorizations associated with parallel processing [2,6,7,12–14], and [15]. Numerous related papers have since followed [1,8,9,3–5,10], and [11]. Ghoshal, Laskar, and Pillone were the first to investigate minimal $k$-rankings from a mathematical standpoint [6,7,13], and [14]. The determination of the rank number and the arank number was shown to be NP-complete [14]. The rank number was explored in [1] where they showed $\chi_r(P_n) = \lfloor \log_2 n \rfloor + 1$. Rank numbers are known for a few other graph families such as cycles, wheels, complete bipartite graphs, and split graphs (see [7] and [4]). Less is known about the arank number. Arank numbers for complete bipartite graphs and split graphs were established in [7]. Kostyuk, Narayan, and Williams [11] showed $\psi_r(P_n) = \lfloor \log_2(n+1) \rfloor + \lfloor \log_2(n+1-(2^{\lfloor \log_2 n \rfloor - 1})) \rfloor$. Recently, the arank number of a cycle was investigated by Kostyuk and Narayan [10].

Throughout the paper $P_n$ denotes the path on $n$ vertices. We use $P_2 \times P_n$ to denote the *Cartesian product* of $P_2$

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and $P_n$. The $k$th power of a path, denoted $P_n^k$, has vertices $v_1, v_2, \ldots, v_n$ and edges $(v_i, v_j)$ for all $|i - j| \leqslant k$.

In this paper we build upon known results and establish new rank numbers. Our main results are stated in the first two theorems. The first theorem gives the rank number of $P_2 \times P_n$ and a relation to the arank number of a path.

**Theorem 1.** $\chi_r(P_2 \times P_n) = \psi_r(P_n) + 1$.

An interesting result follows by a simple extension. We prove that the minimum $k$ in a minimal $k$-ranking of $P_n^2$ is the maximum $k$ in a minimal $k$-ranking of $P_n$.

**Theorem 2.** *We have* $\chi_r(P_n^2) = \psi_r(P_n)$.

## 2. The rank number of $P_n^2$

We begin by restating two elementary results of Ghoshal et al. [7].

**Lemma 3.** *In any minimal ranking of a connected graph G the highest label must be unique.*

**Proof.** Suppose there exist two vertices $u$ and $v$ which both have the highest label $k$. Then any path between $u$ and $v$ will not contain a vertex with a higher label. This is a contradiction. □

The following lemma gives a monotonicity result involving the rank number.

**Lemma 4.** *Let $H$ be a subgraph of a graph $G$. Then* $\chi_r(H) \leqslant \chi_r(G)$.

**Proof.** The proof is straightforward. Suppose $\chi_r(H) > \chi_r(G)$. Then we could relabel the vertices of $H$ using the corresponding labels used in the ranking of $G$. This produces a ranking with fewer labels, and hence a contradiction. □

### 2.1. The ladder graph $L_n$

Before investigating the rank number of $P_n^2$ we describe a family of graphs built using the *Cartesian product*.

**Definition 5.** The *Cartesian product* of $G$ and $H$ written $G \times H$ is the graph with vertex set $v(G) \times V(H)$ specified by putting $\{u, v\}$ adjacent to $(u', v')$ if and only if (i) $u = u'$ and $(v, v') \in E(H)$ or (ii) $v = v'$ and $(u, u') \in E(G)$.

We define the ladder graph, $L_n = P_2 \times P_n$ as shown in Fig. 1.

**Fig. 1.** The ladder graph $L_n = P_2 \times P_n$.

**Example 6.** We give labelings showing $\chi_r(L_n)$ for $n \leqslant 4$ in Fig. 2.

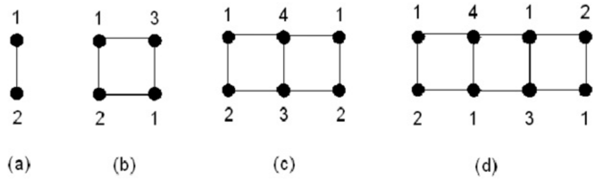


**Fig. 2.** Minimal $\chi_r$-rankings of small ladders.

All four cases are easy to prove. The strategy in all of the cases will be to establish a lower bound and then construct a $k$-ranking where $k$ equals that lower bound. We have $\chi_r(L_1) = 2$ since we cannot have two adjacent vertices labeled with a 1 and a minimal 2-ranking is shown in Fig. 2(a). We next consider the case where $n \geqslant 2$. If $\chi_r(L_2) = 2$ then we can apply Lemma 3 to conclude that the vertex labeled 2 must be unique, and the remaining labels are all labeled 1. This is impossible since we would be forced to have adjacent vertices both with labels of 1. Hence $\chi_r(L_2) \geqslant 3$, and since we have a minimal 3-ranking in Fig. 2(b) it follows that $\chi_r(L_2) = 3$. Next we consider the third case where $n = 3$. By Lemma 4 we have $\chi_r(L_3) \geqslant \chi_r(L_2) \geqslant 3$. If $\chi_r(L_3) = 3$ then we can conclude by Lemma 3 that the vertex labeled 3 must be unique. This means that the remaining five vertices would all receive labels that are either 2 or 1. We are guaranteed to have either two adjacent vertices labeled 1 or a $P_3$ with vertices labeled (in order) 2, 1 and 2. Both cases violate the ranking condition, and hence $\chi_r(L_3) \geqslant 4$. Since a minimal 4-ranking is shown in Fig. 2(c) it follows that $\chi_r(L_3) = 4$. Finally for the last case we apply Lemma 4 to conclude that $\chi_r(L_4) \geqslant \chi_r(L_3) = 4$. The given minimal 4-ranking in Fig. 2(d) shows that this bound is tight.

We now seek to extend the minimal $\chi_r$-rankings of small ladders to minimal $\chi_r$-rankings of large ladders. In our construction we will start with two copies of $L_s$ and join them with a bridge that consists of either a single ‘vertical edge’ or a pair of ‘vertical edges’ along with an appropriate set of edges that connects the pieces to form a larger ladder. These are demonstrated in Figs. 3 and 4. We call these connecting subgraphs, 1-*bridges* and 2-*bridges*, respectively. We next present two lemmas which will be used to build minimal $\chi_r$-rankings of large ladders. In Lemma 7 we let $G = L_{2s+1}$ and in Lemma 8 we let $G = L_{2s+2}$. Each of the lemmas will show that $\chi_r(G) \geqslant \chi_r(L_s) + 2$. The idea is to first note that since $L_s$ is a subgraph of $G$ we can apply Lemma 4 to conclude that $\chi_r(G) \geqslant \chi_r(L_s)$. The insertion of the bridge adds either two or four vertices that push the rank number up by at least 2.

**Lemma 7** *(First Ladder Lemma). Let $G$ be the union of two copies of $L_s$ connected by a* 1-*bridge. Then* $\chi_r(G) \geqslant \chi_r(L_s) + 2$.

**Proof.** Let the two added vertices be labeled $a$ and $b$. We consider cases for different minimal rankings of $G$. We will show in each case there is a vertex with a label greater than or equal to $\chi_r(L_s) + 2$.