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Improved neural dynamics for online Sylvester equations solving

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ABSTRACT

A novel implicit dynamic system together with its electronic implementation is firstly proposed and investigated for online solution of Sylvester equations. In view of the success of recently-proposed Zhang implicit dynamics, our proposed model is also designed in the implicit-dynamical fashion. Compared to the existing neural dynamics, i.e., conventional gradient explicit dynamics and Zhang implicit dynamics, our implicit dynamic system can achieve superior global exponential convergence performance. Computer simulation results demonstrate theoretical analysis of our proposed dynamic model for real-time solution of Sylvester equations.

1. Introduction

Sylvester equations, a special type of linear matrix equations, can be presented as the following:

$$\boldsymbol{A}\boldsymbol{X}(t)+\boldsymbol{X}(t)\boldsymbol{B}+\boldsymbol{C}=\boldsymbol{0}, \tag{1}$$

where m-dimension square matrix $\boldsymbol{A}$, n-dimension square matrix $\boldsymbol{B}$, matrix $\boldsymbol{C}\in\mathbb{R}^{m\times n}$, and null matrix $\boldsymbol{0}\in\mathbb{R}^{m\times n}$ are coefficients. Unknown matrix $\boldsymbol{X}(t)\in\mathbb{R}^{m\times n}$ is to be solved. Evidently, when $\boldsymbol{B}=\boldsymbol{A}^{\mathrm{T}}$ and $m=n$, the problem reduces to Lyapunov equations [1,2]. For solving online Sylvester Equations effectively and efficiently is a hot and classic research topic in automatic control [3–8].

Most existing techniques for solving Sylvester equations fall into three categories. A classic and straight-forward solution for solving Sylvester equations is matrix-inversion based [3]. It is well known that the computational cost of matrix inversion is proportional to the cubic of the matrix's dimension, which could be less efficient for large-scale or real-time applications. Serial-processing numerical solutions are then proposed for solving Sylvester equations [4–6] in an inverse-free manner, but those methods still suffer expensive computational cost. Owing to its parallel distributed nature and the convenience of circuits realization [7–9], neural dynamic solvers, which have been widely used in control theory [7,10–17], robotic inverse kinematics [9,18–22] and system identification [23–25], can be considered as a powerful alternative.

A novel dynamic system in the form of implicit dynamics is proposed and exploited for online solution of Sylvester equations in this letter. Different from the existing neural dynamics [7,8,11], our such dynamic model is developed in a new descent direction to achieve superior convergence performance. The main contributions and novelties are two fold:

- For the first time, this paper proposes a novel implicit dynamic system as well as its electronic realization for real-time solution of Sylvester equations based on a new descent direction.
- Superior global convergence performance can be achieved by our implicit dynamics (ID) compared to the existing dynamic systems, i.e. classic Gradient-

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based Dynamics (GD) [8] and recently-proposed Zhang Dynamics (ZD) [7,8].

Simulative results can demonstrate the effectiveness of the proposed implicit dynamics and verify better performance over the state-of-the-art dynamic systems.

2. Model formulation

For laying a basis for further theoretical analysis, the following unique solution condition is assumed.

Unique-solution condition Sylvester equations (1) is uniquely solvable, if all the eigenvalues of matrix $\boldsymbol{U}_n \otimes \boldsymbol{A} + \boldsymbol{B}^{\mathrm{T}} \otimes \boldsymbol{U}_m$ are nonzero, where $\boldsymbol{U}_n \in \mathbb{R}^{n\times n}$ and $\boldsymbol{U}_m \in \mathbb{R}^{m\times m}$ denote the unit matrices (or termed, the identity matrix) of appropriate dimensions. Symbol $\otimes$ denotes the Kronecker product; i.e., $\boldsymbol{P} \otimes \boldsymbol{Q}$ is a large matrix made by replacing the ijth entry p_{ij} of $\boldsymbol{P}$ with the matrix $p_{ij}\boldsymbol{Q}$ [27].

Our novel recurrent model in the form of implicit dynamics (i.e., $\boldsymbol{A}\dot{\boldsymbol{X}}(t)+\dot{\boldsymbol{X}}(t)\boldsymbol{B} = \cdots$) is firstly proposed in this paper for real-time solution of Sylvester equations as the following (refer to Appendix A for the derivation of such a model):

$$\begin{aligned}\boldsymbol{A}\dot{\boldsymbol{X}}(t)+\dot{\boldsymbol{X}}(t)\boldsymbol{B} &= -\gamma(\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}}+\boldsymbol{U}_m)(\boldsymbol{A}\boldsymbol{X}(t)+\boldsymbol{X}(t)\boldsymbol{B}+\boldsymbol{C})\\ &\quad -\gamma(\boldsymbol{A}\boldsymbol{X}(t)+\boldsymbol{X}(t)\boldsymbol{B}+\boldsymbol{C})(\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B}+\boldsymbol{U}_n)\\ &\quad -\gamma\boldsymbol{A}(\boldsymbol{A}\boldsymbol{X}(t)+\boldsymbol{X}(t)\boldsymbol{B}+\boldsymbol{C})\boldsymbol{B}^{\mathrm{T}}\\ &\quad -\gamma\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{A}\boldsymbol{X}(t)+\boldsymbol{X}(t)\boldsymbol{B}+\boldsymbol{C})\boldsymbol{B}, \qquad (2)\end{aligned}$$

where design parameter γ is used to adjust the convergence rate and can be selected appropriately for illustrative purpose (e.g. $\gamma = 100$ and $\gamma = 1000$ in our simulation).

For comparative purpose, the existing neural dynamics, i.e. gradient-based dynamics (GD) [8] and recently-proposed Zhang dynamics (ZD) [7] are investigated in this paper. On one hand, gradient-based dynamic model in the form of explicit dynamic form (i.e. $\dot{\boldsymbol{X}}(t) = \cdots$) for online Sylvester equations solving is given as follows (refer to [8, 11] for more details):

$$\begin{aligned}\dot{\boldsymbol{X}}(t) &= -\gamma\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{A}\boldsymbol{X}(t)+\boldsymbol{X}(t)\boldsymbol{B}+\boldsymbol{C})\\ &\quad -\gamma(\boldsymbol{A}\boldsymbol{X}(t)+\boldsymbol{X}(t)\boldsymbol{B}+\boldsymbol{C})\boldsymbol{B}^{\mathrm{T}}. \qquad (3)\end{aligned}$$

On the other hand, by following Zhang et al.'s design method [7,8], so-called Zhang dynamics originally designed for time-varying problems can also be exploited for constant Sylvester equations solving:

$$\boldsymbol{A}\dot{\boldsymbol{X}}(t)+\dot{\boldsymbol{X}}(t)\boldsymbol{B} = -\gamma(\boldsymbol{A}\boldsymbol{X}(t)+\boldsymbol{X}(t)\boldsymbol{B}+\boldsymbol{C}). \qquad (4)$$

Before ending this section, we compare the proposed ID (2) with GD (3) and ZD (4). Our proposed ID (2) is based on a different descent direction rather than the elimination of the scalar-valued norm-based energy function of Gradient dynamics (3) or the elimination of every entry of the vector-valued error function of Zhang dynamics (4) [7,8]. Moreover, ID (2) can take advantages of implicit dynamic systems over explicit systems [26] but also can theoretically achieve superior convergence performance to GD (3) and ZD (4), which will be analysed and verified via computer simulation in the following sections.

3. Electronic realization

It follows from Kronecker-product and vectorization techniques [7,27] that model (2) can be transformed as:

$$\begin{aligned}&(\boldsymbol{U}_n\otimes\boldsymbol{A}+\boldsymbol{B}^{\mathrm{T}}\otimes\boldsymbol{U}_m)\mathrm{vec}\big(\boldsymbol{X}(t)\big)\\ &\quad=-\gamma\big((\boldsymbol{U}_n\otimes\boldsymbol{A}+\boldsymbol{B}^{\mathrm{T}}\otimes\boldsymbol{U}_m)(\boldsymbol{U}_n\otimes\boldsymbol{A}\\ &\qquad+\boldsymbol{B}^{\mathrm{T}}\otimes\boldsymbol{U}_m)^{\mathrm{T}}+2\boldsymbol{U}_{mn}\big)\times\big((\boldsymbol{U}_n\otimes\boldsymbol{A}\\ &\qquad+\boldsymbol{B}^{\mathrm{T}}\otimes\boldsymbol{U}_m)\mathrm{vec}\big(\boldsymbol{X}(t)\big)+\mathrm{vec}(\boldsymbol{C})\big),\end{aligned}$$

where operator $\mathrm{vec}\big(\boldsymbol{X}(t)\big) \in \mathbb{R}^{mn}$ denotes a column vector obtained by stacking all column vectors of $\boldsymbol{X}(t)$ together. Following the above vector-formed dynamics, ith element's dynamic equation ($i = 1, 2, \cdots, \mathrm{N}$, where $\mathrm{N} = mn$ in our case) can be presented as the following:

$$\begin{aligned}\dot{x}_i &= -\gamma\left(\sum_{j=1}^{\mathrm{N}} m_{ij}x_j - v_i\right) - \left(\sum_{j=1, j\neq i}^{\mathrm{N}} k_{ij}\dot{x}_j\right)\\ &\quad + (1-k_{ii})\dot{x}_i, \qquad (5)\end{aligned}$$

where k_{ij}, m_{ij}, and v_i denote the corresponding elements of coefficients $\boldsymbol{K} := \boldsymbol{U}_n\otimes\boldsymbol{A}+\boldsymbol{B}^{\mathrm{T}}\otimes\boldsymbol{U}_m$, $\boldsymbol{M} := (\boldsymbol{K}\boldsymbol{K}^{\mathrm{T}} + 2\boldsymbol{U}_{mn})\boldsymbol{K}$, and $\boldsymbol{v} := (\boldsymbol{K}\boldsymbol{K}^{\mathrm{T}}+2\boldsymbol{U}_{mn})\mathrm{vec}(\boldsymbol{C})$, respectively. The architecture of the implementation of our dynamic system (2) is illustrated in Fig. 1.

4. Convergence analysis

Convergence properties of our dynamic model (2) is investigated and analyzed with two propositions about global convergence by gradient-based and Zhang dynamics for online Sylvester equations solving also presented for comparative purpose.

Proposition 1. *Given coefficient matrices $\boldsymbol{A} \in \mathbb{R}^{m\times m}$ and $\boldsymbol{B} \in \mathbb{R}^{n\times n}$ satisfying the Unique-Solution Condition, state matrix $\boldsymbol{X}(t) \in \mathbb{R}^{m\times n}$ of gradient-based dynamics (3), starting from any initial state $\boldsymbol{X}(0) \in \mathbb{R}^{m\times n}$, will globally exponentially converge to theoretical solution $\boldsymbol{X}^*$. In addition, the exponential convergence rate is the product of γ and the minimum eigenvalue $\alpha > 0$ of $(\boldsymbol{U}_n\otimes\boldsymbol{A}+\boldsymbol{B}^{\mathrm{T}}\otimes\boldsymbol{U}_m)(\boldsymbol{U}_n\otimes\boldsymbol{A}+\boldsymbol{B}^{\mathrm{T}}\otimes\boldsymbol{U}_m)^{\mathrm{T}}$.*

Proposition 2. *Given coefficient matrices $\boldsymbol{A} \in \mathbb{R}^{m\times m}$ and $\boldsymbol{B} \in \mathbb{R}^{n\times n}$ satisfying the Unique-Solution Condition, state matrix $\boldsymbol{X}(t) \in \mathbb{R}^{m\times n}$ of Zhang dynamics (4), starting from any initial state $\boldsymbol{X}(0) \in \mathbb{R}^{m\times n}$, will globally exponentially converge to the unique exact solution $\boldsymbol{X}^*$ with the exponential convergence rate γ.*

Theorem 1. *Given coefficient matrices $\boldsymbol{A} \in \mathbb{R}^{m\times m}$ and $\boldsymbol{B} \in \mathbb{R}^{n\times n}$ satisfying the Unique-Solution Condition, state matrix $\boldsymbol{X}(t) \in \mathbb{R}^{m\times n}$ of our proposed dynamics (2), starting from any initial state $\boldsymbol{X}(0) \in \mathbb{R}^{m\times n}$, will globally exponentially converge*