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Testing whether the uniform distribution is a stationary distribution

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A R T I C L E I N F O

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A B S T R A C T

A random walk on a directed graph generates a Markov chain on the vertices of the graph. An important question that often arises in the context of Markov chains is, whether the uniform distribution on the vertices of the graph is a stationary distribution. A stationary distribution of a Markov chain is a global property of the graph. This leads to the belief that whether a particular distribution is a stationary distribution of a Markov chain depends on the global property of that Markov chain. In this paper for a directed graph whose underlying undirected graph is regular, we prove that whether the uniform distribution on the vertices of the graph is a stationary distribution, depends on a local property of the graph, namely if (u, v) is a directed edge, then out-degree(u) is equal to in-degree(v).
This result also has an application to the problem of testing whether a given distribution is uniform or "far" from being uniform. If the distribution is the stationary distribution of the lazy random walk on a directed graph and the graph is given as an input, then how many bits (orientations) of the input graph does one need to query in order to decide whether the distribution is uniform or "far"[2] from it? This is a problem of graph property testing, and we consider this problem in the orientation model. We reduce this problem to testing Eulerianity in the orientation model.

1. Introduction

Spectral properties of undirected graphs have been well studied and well understood [2]. However, there has been less success in the study of the same in the case of directed graphs, possibly due the non-symmetric structure associated with its adjacency matrix. In this work, we attempt to understand the spectral properties of Markov chains obtained by a random walk on a directed graph. Markov chains are one of the most important structures in Theoretical Computer Science. The most significant characteristics of a Markov chain are its stationary distribution and mixing time. It is an interesting problem to test if a particular distribution is a stationary distribution of a given Markov chain. Since a stationary distribution of a Markov chain is a global property of the graph, this leads to the belief that whether a particular distribution is a stationary distribution of a Markov chain depends on its global structure. In this paper, we focus on the Markov chain obtained by a random walk on a directed graph. We prove contrary to aforementioned belief that if the graph is regular, then whether the uniform distribution is a stationary

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[1] This work done when the author was pursuing his PhD from Chennai Mathematical Institute.

[2] Here, farness does not imply any statistical distance between the distributions. Rather, it specifies the distance between the orientations – i.e. the minimum number of edges that need to be reoriented such that the stationary distribution obtained by a random walk on the resulting graph (obtained after reorientation of edges) is uniform.

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distribution depends on a local property of the graph. The following theorem, which is the main result of this paper, is a statement about that local property.

Theorem 1. *Let $\overrightarrow{G} = (V, \overrightarrow{E})$ be a directed graph such that the total degree (i.e., Indegree(v) + Outdegree(v)) for every vertex $v \in V$ is the same. Then the uniform distribution on the vertices of $\overrightarrow{G}$ is a stationary distribution of the Markov chain (generated by a random walk on $\overrightarrow{G}$) if and only if the graph has the following properties:*

1. *for all $v \in V$, Indegree(v) $\neq 0$ and Outdegree(v) $\neq 0$,*
2. *for all $(u, v) \in \overrightarrow{E}$, Outdegree(u) = Indegree(v).*

As an application of this result, we design a testing algorithm to test whether the uniform distribution is the stationary distribution of the Markov chain generated by a *lazy random walk*[3] on $\overrightarrow{G}$.

1.1. Application to property testing of distributions

In property testing, the goal is to look at a very small fraction of the input and decide whether the input has a certain property, or it is "far" from satisfying the property. For a given distance parameter $0 < \epsilon < 1$, we say that the input is an ϵ-far from satisfying the property if one has to change at least ϵ fraction of the input to make the input satisfy the property. Theorem 1 also has an application to the problem of testing whether a given distribution is uniform or "far" from being uniform. More precisely, if the distribution is the stationary distribution of the lazy random walk on a directed graph and the graph is given as an input, then how many bits of the input graph does one need to query in order to decide whether the distribution is uniform or "far" from it? We consider this problem in the **orientation model** (see [5]). In this model, the underlying undirected graph $G = (V, E)$ is known in advance, and the orientation of the edges has to be queried. The graph is said to be "ϵ-far" from satisfying the property P if one has to reorient at least an ϵ fraction of the edges to make the graph satisfy the property. We reduced this problem to testing Eulerianity in the orientation model. We use the results of [4] to obtain an algorithm that incurs sublinear cost for the above problem. We present this part of our result in Section 4.

2. Preliminaries

2.1. Graph notations

Throughout the paper, we will be dealing with directed graphs (possibly with multiple edges between any two vertices) in which each edge is directed only in one direction. To avoid confusion, we will call them **oriented graphs** because each edge is oriented and is not bidirectional. We will denote the oriented graph by $\overrightarrow{G} = (V, \overrightarrow{E})$, and the underlying undirected graph (that is, when the directions on the edges are removed) by $G = (V, E)$. For a vertex $v \in V$, the in-degree and the out-degree of v in $\overrightarrow{G}$ are denoted by $d^-(v)$ and $d^+(v)$ respectively. An oriented graph $\overrightarrow{G} = (V, \overrightarrow{E})$ is called a *degree-*Δ oriented graph if for all $v \in V$, $d^-(v) + d^+(v) = \Delta$. In this paper, we will be focusing on *degree-*Δ oriented graphs.

2.2. Markov chains preliminaries

Fact 2. *A Markov chain is a stochastic process on a set of states given by a transition matrix. Let S be the set of states with $|S| = n$. Then, the transition matrix T is an $n \times n$ matrix with entries from the positive reals; the rows and columns are indexed by the states; the (u, v)-th entry $T_{u,v}$ of the matrix denotes the probability of transition from state u to state v. Since T is stochastic, $\sum_v T_{u,v}$ must be 1. A distribution $\mu : S \to \mathbb{R}^+$ on the vertices is said to be stationary if for all vertices v,*

$$\sum_v \mu(u) T_{u,v} = \mu(v).$$

Fact 3. *If $\overrightarrow{G}$ is an oriented graph then a random walk on $\overrightarrow{G}$ defines a Markov chain, where, the states are the vertices of the graph; the probability to traverse an edge (u, v) is given by the quantity $p_{u,v} = \frac{1}{d^+(u)}$; and hence, the transition probability $T_{u,v}$ from vertex u to vertex v is $p_{u,v}$ times the number of edges between u and v. The uniform distribution on the vertices of $\overrightarrow{G}$ is a stationary distribution for this Markov chain if and only if for all $v \in V$,*

$$\sum_{u:(u,v)\in\overrightarrow{E}} p_{u,v} = 1 = \sum_{w:(v,w)\in\overrightarrow{E}} p_{v,w}.$$

Note 4. In this paper, we will only consider Markov chains that arise from random walks on $\overrightarrow{G}$, where $\overrightarrow{G}$ is an oriented graph.

3. Structure of graphs with uniform stationary distribution

The following theorem is a rephrasing of Theorem 1.

Theorem 5. *Let $\overrightarrow{G} = (V, \overrightarrow{E})$ be a degree-Δ oriented graph, then the uniform distribution on the vertices of $\overrightarrow{G}$ is a stationary distribution of the Markov chain if and only if for all $v \in V$, both $d^-(v), d^+(v) \neq 0$ and for all $(u, v) \in \overrightarrow{E}$,*

$$d^+(u) = d^-(v).$$

Proof. First of all, recall that the uniform distribution is a stationary distribution for $\overrightarrow{G}$, if and only if for all $v \in V$,

$$\sum_{u:(u,v)\in\overrightarrow{E}} p_{u,v} = 1 = \sum_{w:(v,w)\in\overrightarrow{E}} p_{v,w},$$

[3] A lazy random walk (starting from a particular vertex) on a directed graph is a random walk in which at each time step, the walk stays where it is with probability $\frac{1}{2}$ or moves according to the usual random walk. Moreover, it converges to a unique stationary distribution.