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Enumeration of balanced finite group valued functions on directed graphs

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ABSTRACT

A group valued function on a graph is called balanced if the product of its values along any cycle is equal to the identity element of the group. We compute the number of balanced functions from the set of edges and vertices of a directed graph to a finite group considering two cases: when we are allowed to walk against the direction of an edge and when we are not allowed to walk against the edge direction. In the first case it appears that the number of balanced functions on edges and vertices depends on whether or not the graph is bipartite, while in the second case this number depends on the number of strong connected components of the graph.

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1. Introduction

Let G be a finite group with the group operation denoted by $\cdot$ and the identity element denoted by 1. Let Γ be a graph. Roughly speaking, a G-valued function f on vertices and/or edges of Γ is called *balanced* if the product of its values along any cycle of Γ is 1.

The study of balanced functions can be conducted in three cases:

1. The graph Γ is directed with the set of vertices V and the set of directed edges E. When traveling between the vertices, we are allowed to travel with or against the direction of the edges. The value of a function f on $\bar{e}$, which represents traveling the edge e against its direction, is equal to $(f(e))^{-1}$. In this context, when the function is defined on edges only, the pair (Γ, f) is called a network or a directed network. In this paper we shall call this *the flexible case*, meaning that the direction of an edge does not forbid us to walk against it. In the case when the group G is Abelian (in particular $\mathbb{Z}$ or $\mathbb{R}$) the notion of balanced functions on edges for the flexible case, for functions taking values only on the edges, is introduced in the literature under different names. Thus, for example, in [3] the set of such functions is exactly $Im(d)$, and in [14], in somewhat different language, that set is referred to as the set of consistent graphs. In [19] such functions have been introduced under the name "color-coboundaries". Also they appear in literature under the name "tensions". They have been extensively studied, recent examples include [5,6,8,20,13]. In [7] balanced $\mathbb{C}$-valued functions on edges appear in a certain connection with geometric representations of the Coxeter group associated with a graph. When the group G is Abelian, the set of balanced functions forms a group in a natural way, the structure of this group was studied in [10]. In a rather common terminology

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introduced by Zaslavsky, [21], a pair of a graph and such a function on edges of a graph is called a gain graph.

2. The graph Γ is directed with the set of vertices V and the set of directed edges E, but we are only allowed to travel with the direction of the edges. In this paper we shall call this *the rigid case*. When f takes values only on the edges then in some literature, following Serre, [18], the flexible case is described as a particular instance of the rigid case by introducing the set $\mathbb{E}$ as the new set of directed edges of G (the cardinality of $\mathbb{E}$ is twice that of E), denoting by $\bar{e} \in \mathbb{E}$ the inverse of the directed edge $e \in E$ and requiring $f(\bar{e}) = (f(e))^{-1}$, [3,18].
3. The graph Γ is undirected. The value of a function f on an edge e does not depend on the direction of the travel on e. The case of balanced functions $f : E \to \mathbb{R}$ is studied in [4], where these functions are called "cycle-vanishing edge valuations". Balanced $\mathbb{Z}_2$-valued functions on vertices only were studied and characterized under the name of consistent marked (vertex-signed) graphs in [1,2,15,17] and several other works. In [16] some relations between consistent marked graphs and balanced signed graphs are studied. For an Abelian group A, the case of balanced functions $f : V \bigcup E \to A$ is first introduced and studied in [12]. The group structure of the groups of balanced functions on edges, and balanced functions on vertices and edges of an undirected graph with values in an Abelian group is studied in [9].

Notice that if we take functions with values in a non-Abelian group then the sets of the balanced function on edges and balanced functions on vertices and edges do not inherit any natural group structure. However, to find the number of these functions for directed graphs when the non-Abelian group is finite is possible, and we do it in the present paper.

In what follows we say that a directed graph is *weakly connected* if its underlying undirected graph is connected.

For the basics of Graph Theory we refer to [11].

2. The flexible case

Let $\Gamma = (V, E)$ be a weakly connected directed graph, possibly with loops and multiple edges. Let $v, w \in V$ be two vertices connected by an edge e; v is the origin of e and w is the endpoint of e. For $e \in E$ denote by $\bar{e}$ the same edge as e but taken in the opposite direction. Thus $\bar{e}$ goes from w to v. Let $\mathbb{E} = \{e, \bar{e} \mid e \in E\}$.

Definition 2.1. A path P from a vertex x to a vertex y is an alternating sequence $v_1, e_1, v_2, e_2, \ldots, v_n, e_n$ of vertices from V and different edges from $\mathbb{E}$ such that $v_1 = x$ and each e_j, for $j = 1, \ldots, n-1$, goes from v_j to v_{j+1} and e_n goes from v_n to y. We permit the same edge e to appear in a path twice – one time along and one time against its direction, since this is regarded as using two different edges from $\mathbb{E}$.

We require our graphs to be weakly connected. Namely, any two different vertices of our graph Γ can be connected by a path.

Definition 2.2. A path P from a vertex x to itself is called a cycle.

We permit the trivial cycle, which is the empty sequence containing no vertices and no edges.

Definition 2.3. The length of a cycle is the number of its edges.

Definition 2.4. A function $f : \mathbb{E} \to G$ such that $f(\bar{e}) = (f(e))^{-1}$ is called *balanced* if the product $f(e_1) \cdot f(e_2) \cdots f(e_n)$ of the values of f over all the edges of any cycle of Γ equals to 1.

Definition 2.5. The set of all balanced functions $f : \mathbb{E} \to G$ is denoted by $\mathcal{FE}(\mathbb{E}, G)$.

Definition 2.6. A function $h : V \bigcup \mathbb{E} \to G$, which takes both vertices and edges of Γ to some elements of G, is called balanced if $h(\bar{e}) = (h(e))^{-1}$ and the product of its values $h(v_1) \cdot h(e_1) \cdot h(v_2) \cdot h(e_2) \cdots h(v_n) \cdot h(e_n)$ along any cycle of Γ is 1.

Definition 2.7. The set of all balanced functions $h : V \bigcup \mathbb{E} \to G$ is denoted by $\mathcal{FU}(\Gamma, G)$.

Definition 2.8. The set of all involutions of G is denoted G_2. I.e., $G_2 = \left\{a \in G \mid a^2 = 1\right\}$.

The set $\mathcal{FE}(\mathbb{E}, G)$ is well understood and the following fact is well known.

Proposition 2.9. *For a finite group G, the cardinality of the set $\mathcal{FE}(\mathbb{E}, G)$ is equal to $|G|^{|V|-1}$.*

Proof. Select a vertex v and notice that the number of all functions $g : V \to G$ such that $g(v) = 1$ is equal to $|G|^{|V|-1}$. Consider the following bijection between the set of all G-valued functions g on V with $g(v) = 1$ and the set $\mathcal{FE}(\mathbb{E}, G)$. For any such g, since each edge $e \in \mathbb{E}$ goes from some vertex x to some vertex y, we define $f(e) = (g(x))^{-1} \cdot g(y)$. A straightforward calculation shows that $f \in \mathcal{FE}(\mathbb{E}, G)$. In the other direction of the bijection, for $f \in \mathcal{FE}(\mathbb{E}, G)$ we inductively construct the function g as follows: we set $g(v) = 1$; if $g(u)$ has been defined for a vertex u then for every vertex w, for which there exists some edge e from u to w, we define $g(w) = g(u) \cdot f(e)$. Since $f \in \mathcal{FE}(\mathbb{E}, G)$, any two calculations of the value of g on any vertex u will produce the same result. The weak connectivity implies that every vertex indeed receives a value, and thus, our g is well-defined. $\square$

Our main result is the following.

Theorem 2.10. *Let $\Gamma = (V, E)$ be a weakly connected directed graph and Γ' be its underlying undirected graph. Then:*

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