Information Processing Letters 115 (2015) 817–821

Contents lists available at ScienceDirect

Information Processing Letters

www.elsevier.com/locate/ipl

Subnetwork preclusion for bubble-sort networks

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ARTICLE INFO

Article history:
Received 27 October 2014
Received in revised form 1 May 2015
Accepted 15 June 2015
Available online 19 June 2015
Communicated by M. Yamashita

Keywords:
Multiprocessor systems
Interconnection networks
Subnetwork preclusion
Bubble-sort graphs

ABSTRACT

For two positive integers n and m with $n > m$, the G_{n-m} preclusion node (resp. link) number $F_m(G_n)$ (resp. $f_m(G_n)$) of an n-dimensional interconnection network G_n is the minimum number of nodes (resp. links), if any, whose deletion results in a network with no subnetwork isomorphic to G_{n-m}. The n-dimensional bubble-sort network B_n is one of the most attractive interconnection networks for multiprocessor systems. In this paper, we prove that $F_2(B_n) = f_2(B_n) = n(n-1)$ for $n \geq 6$.

1. Introduction

The interconnection network plays an important role in large-scale multiprocessor systems, and is usually represented by an undirected graph $G = (V, E)$, where nodes in V correspond to processors, and edges in E correspond to communication links. In a real system, failures of components are inevitable. Thus, fault tolerance of interconnection networks has become an important issue. Fault tolerance of interconnection networks is usually measured by how much of the network structure is preserved in the presence of a given number of component failures. Obviously, in the presence of component failures, the entire interconnection network is not available. Under this consideration, Becker and Simon [3] proposed a problem: if the network contained a given number of faulty processors or links, then what the maximum number of dimensions that would be lost is. About ten years later, Latifi [9] proposed a similar question that how large of a subnetwork is still available in the network in the presence of component failures. Motivated by the above problems and the concept of matching preclusion [5,6], we presented the following:

Let G_n be an n-dimensional recursive interconnection network with $n \geq 2$. Given an integer $2 \leq j \leq n$, let G_i be the subnetwork, smaller network but with the same topological properties as the original one, of G_j for $1 \leq i \leq j-1$. Given a positive integer m with $m < n$, *the* G_{n-m} *preclusion node number* $F_m(G_n)$ (resp. G_{n-m} *preclusion link number* $f_m(G_n)$) of G_n is the minimum number of nodes (resp. links), if any, whose deletion results in a network with no subnetwork isomorphic to G_{n-m}.

Becker and Simon [3] studied the subnetwork preclusion problem in the hypercube network H_n and determined $f_m(H_n)$ for some m's. Latifi [9] investigated the problem in the star network S_n and mainly determined $f_2(S_n)$ when n is prime. Later, Latifi et al. [10] proved that $f_1(S_n) = n + 2$ and gave an upper bound on $f_2(S_n)$ for $n \geq 4$, with complexity of $O(n^3)$. Walker and Latifi [12] improved the bound on $f_m(S_n)$ and gave a relationship between $f_m(S_n)$ and $F_m(S_n)$. Recently, Wang and Yang [13] explored the problem in the bubble-sort network B_n. They determined $F_1(B_n)$ and $f_1(B_n)$, and presented a nontrivial upper bound on $F_2(B_n)$ and $f_2(B_n)$. Subsequently, Wang et al. [14] investigated the problem in k-ary n-cubes and determined the k-ary $(n-1)$-cube preclusion

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http://dx.doi.org/10.1016/j.ipl.2015.06.011
0020-0190/

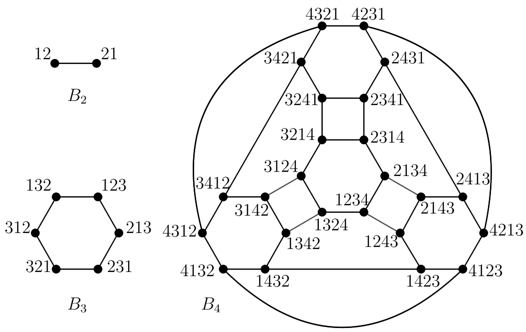

Fig. 1. The bubble-sort graphs B_2, B_3 and B_4.

node number and gave a nontrivial upper bound on k-ary $(n-2)$-cube preclusion node number of k-ary n-cubes. Wang and Feng [15] investigated the problem in the arrangement graph $A_{n,k}$, which is a generalized network of S_n. They derived the $A_{n-1,k-1}$ preclusion node number, and gave a nontrivial upper bound on the $A_{n-m,k-m}$ preclusion node number of $A_{n,k}$ for $m \geq 2$.

The bubble-sort graph B_n is an attractive interconnection network with many good properties such as node-symmetric, $(n-1)$-regular and bipartite [1,2,7,8,11–13,16]. In [13], one of the main results is shown as follows.

Theorem 1. *(See [13].) Let B_n be an n-dimensional bubble-sort graph. Then $n(n-1) \leq F_2(B_n) \leq f_2(B_n) \leq n(2n-3)$ for $n \geq 6$.*

In this paper, we will improve the above result and prove that $F_2(B_n) = f_2(B_n) = n(n-1)$ for $n \geq 6$.

2. Preliminaries

To be more self sufficient and self contained, we reiterate the terminology and notation used in [13]. For the graph-theoretical terminology and notation not defined here we follow [4]. Let $N_0 = \emptyset$ and let N_n be the set $\{1, 2, \ldots, n\}$ for an arbitrary integer $n \geq 1$.

The *bubble-sort graph*, B_n, $n \geq 1$, is an undirected graph consisting of $n!$ nodes each of which has the form $x = x_1x_2 \ldots x_n$, where $1 \leq x_i \leq n$ and $x_i \neq x_j$ for distinct $1 \leq i, j \leq n$. Two nodes are jointed with an *i-link* if and only if the label of one can be obtained from the label of the other by swapping the ith digit and the $(i+1)$th digit where $i \in N_{n-1}$. B_n has a recursive structure. More specifically, B_n contains n disjoint sub-bubble-sort graphs B_{n-1}. There are exactly two ways to partition B_n into n disjoint B_{n-1}'s when $n \geq 3$. This is done by removing all 1-links (or $(n-1)$-links) in B_n. The bubble-sort graphs B_2, B_3 and B_4 are shown in Fig. 1.

Given an integer $n \geq 1$, let Λ_n be the symbol set $\{0, 1, \ldots, n, X\}$, where X denotes a *don't care* symbol. Let $k \in \{0, 1, \ldots, n-1\}$ and let $a_1, a_2, \ldots, a_k$ be pairwise distinct symbols in N_n, where if $k = 0$ no symbol is chosen in N_n. For any integer $0 \leq i \leq k$, let $M_{k,i} = \{a_1a_2 \ldots a_i b_1 b_2 \ldots b_{n-k} a_{i+1}a_{i+2} \ldots a_k : b_1, b_2, \ldots, b_{n-k} \in N_n \setminus \{a_1, a_2, \ldots, a_k\}$ are pairwise distinct$\}$, where $a_1a_2 \ldots a_i$ is an empty string if $i = 0$, and $a_{i+1}a_{i+2} \ldots a_k$ is an empty string if $i = k$. For example, $M_{0,0} = \{b_1b_2 \ldots b_n : b_1, b_2, \ldots, b_n \in N_n$ are pairwise distinct$\}$. Obviously, the subgraph of B_n induced by $M_{k,i}$ is isomorphic to B_{n-k}. For the convenience of representation, for any integers i, k with $0 \leq k \leq n-1$ and $0 \leq i \leq k$, we denote by an n-length string $a_1a_2 \ldots a_i\ X^{n-k} a_{i+1}a_{i+2} \ldots a_k$ the subgraph induced by $M_{k,i}$ in B_n. Note that $a_1a_2 \ldots a_i X^{n-k} a_{i+1}a_{i+2} \ldots a_k$ is just $X^{n-k} a_1a_2 \ldots a_k$ when $i = 0$, and $a_1a_2 \ldots a_i X^{n-k} a_{i+1} a_{i+2} \ldots a_k$ is just $a_1a_2 \ldots a_k X^{n-k}$ when $i = k$. For example, $X^3 1$ and $1X^3$ denote the B_3's induced by $\{2341, 2431, 4231, 4321, 3421, 3241\}$ and $\{1234, 1324, 1342, 1432, 1423, 1234\}$, respectively. Note that for any integer $0 \leq i \leq n-2$, $a_1a_2 \ldots a_i XX a_{i+1}a_{i+2} \ldots a_{n-2}$ denotes a B_2 which has exactly one link $(a_1a_2 \ldots a_i b_1 b_2 a_{i+1}a_{i+2} \ldots a_{n-2}, a_1a_2 \ldots a_i b_2 b_1 a_{i+1}a_{i+2} \ldots a_{n-2})$, where $\{b_1, b_2\} = N_n \setminus \{a_1, a_2, \ldots, a_{n-2}\}$. We shall not distinguish between the graph B_2 and its link. Therefore, we often refer to the graph $a_1a_2 \ldots a_i XX a_{i+1}a_{i+2} \ldots a_{n-2}$ as its link $(a_1a_2 \ldots a_i b_1 b_2 a_{i+1} a_{i+2} \ldots a_{n-2}, a_1a_2 \ldots a_i b_2 b_1 a_{i+1}a_{i+2} \ldots a_{n-2})$.

In fact, given an integer $k \in \{0, 1, \ldots, n-1\}$, a B_{n-k} in B_n can be uniquely labeled by a string of symbols in Λ_n, i.e., $a_1a_2 \ldots a_i X^{n-k} a_{i+1}a_{i+2} \ldots a_k$, where $a_1, a_2, \ldots, a_k$ are pairwise distinct symbols in N_n and $i \in \{0, 1, \ldots, k\}$. The result has been proved in [13].

Two B_{n-k}'s in B_n are *distinct* if their node sets are different and *disjoint* if they have no common node. The following will be used in the proof of Lemma 2 in Section 3.

Lemma 1. *(See [13].) Given two integers $n \geq 1$ and $k \in \{0, 1, \ldots, n-1\}$, there are $(k+1)!\binom{n}{k}$ distinct B_{n-k}'s in B_n, where the term $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ denotes the number of ways to pick k objects out of n objects.*

3. The B_{n-2} preclusion number of B_n

In this section, we are interested in exploring the B_{n-2} preclusion problem in B_n and we will prove that $F_2(B_n) = f_2(B_n) = n(n-1)$ for $n \geq 6$. Let us begin with an important lemma.

Lemma 2. *Given an integer $n \geq 6$, let $Q_n = \{(x, y) : x, y \in N_n$ and $x \neq y\}$. If there is a bijection Ψ_n from Q_n to itself such that $\{x_1, x_2\} \cap \{y_1, y_2\} = \emptyset$ and $M_n = Q_n$, where $(y_1, y_2) = \Psi_n(x_1, x_2)$ and $M_n = \{(x_1, y_2) : (y_1, y_2) = \Psi_n(x_1, x_2)\}$, then $F_2(B_n) = f_2(B_n) = n(n-1)$.*

Proof. Theorem 1 implies that $f_2(B_n) \geq F_2(B_n) \geq n(n-1)$. In the following, it suffices to prove that $f_2(B_n) \leq n(n-1)$.

Lemma 1 implies that there are $(2+1)!\binom{n}{2} = 3n(n-1)$ distinct B_{n-2}'s in B_n, which can be divided into three disjoint sets A_0, A_1 and A_2, where

$$A_0 = \{X^{n-2} a_1 a_2 : a_1, a_2 \in N_n \text{ and } a_1 \neq a_2\},$$

$$A_1 = \{a_1 X^{n-2} a_2 : a_1, a_2 \in N_n \text{ and } a_1 \neq a_2\},$$

$$A_2 = \{a_1 a_2 X^{n-2} : a_1, a_2 \in N_n \text{ and } a_1 \neq a_2\}.$$

Since there is a bijection Ψ_n from Q_n to itself such that $\{x_1, x_2\} \cap \{y_1, y_2\} = \emptyset$, where $(y_1, y_2) = \Psi_n(x_1, x_2)$, the three B_{n-2}'s (i.e., $X^{n-2} y_1 y_2$ in A_0, $x_1 X^{n-2} y_2$ in A_1 and