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# Reducing the diameter of a unit disk graph *via* node addition

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ABSTRACT

This paper addresses a hop-constrained graph design optimization problem which is related to efficiency and reliability issues of communication protocols in wireless networks. In particular, we study the problem of adding a minimum size set of points to a given unit disk graph in such a way that in the resulting graph any two original points have hop-distance at most a given bound $D$. After having proved the hardness of the problem, we propose two different bi-criteria algorithms that, conjunctively, provide logarithmic approximation ratio on both criteria. We remark that our first algorithm, while unable to provide any approximation guarantee in the general case, does yield an $(O(1), O(1))$-approximation for a wide set of instances.



## 1. Introduction

In this paper, we study a basic network design problem in which we are given a set $P$ of $n$ points in the Euclidean plane and a hop-constraint requirement $D$. The goal is to find a minimum cardinality set $S$ of points to be added to $P$ so that every pair of nodes in $P$ is connected by a path of at most $D$ nodes in $P \cup S$ at distance at most 1 from each other. This problem arises in the wireless networks setting. We briefly recall that a wireless network consists of a set of radio stations connected by wireless links such that two stations are connected by a link if they are within the transmission range of each other. In particular, we are concerned with *homogeneous* wireless networks, in which all stations transmit with the same power so determining bidirectional links. The underlying graph theoretical model is the *unit disk graph*.

The hop constraint limits the maximum number of links in the communication paths between any pair of nodes and, hence, it is closely related to restricting the maximum delay transmission time of several fundamental communication protocols.

Reliability is a second remarkable issue concerned with the hop constraint. Assume that, in a communication network, link faults happen with probability $p$ and that all faults occur independently. Then, the probability that a multi-hop transmission fails exponentially increases with the number of hops.

Summarizing, a fixed bound on the maximum number of hops is sometimes a necessary constraint in order to achieve fast and reliable communication protocols. For further motivations in studying hop-constrained graph design problems see [1–5].

There is also a large body of work about the non-geometric, graph theoretic counterpart of the problem, namely, the problem of adding edges to a given graph so as to minimize the resulting diameter. This last problem arises in practical applications like telecommunication networks and airplane flight scheduling [6,7], and it received a lot of attention in the graph theory community (see [8–13]). The state of the art about approximability and non-approximability results for the problem can be found in [14].

Turning our attention back to the wireless network scenario, this paper is a first step in approaching the problem of decreasing the diameter of a geometric graph by adding to it as few nodes as possible. After having provided the

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necessary preliminary definitions in Section 2, we prove the hardness of the problem in Section 3. Then, we consider bi-criteria approximation algorithms. In Section 4 we exploit a result in [15] to derive a polynomial-time algorithm computing a set $S$ of additional nodes such that every pair of nodes in $P$ is connected by a path in $P \cup S$ of at most $\beta D$ nodes at distance at most 1 from each other, for some constant $\beta > 1$. Furthermore, the number of added points is a constant factor away from the size of a minimal set satisfying the hop constraint $D$ up to an additive term of $O(n)$. This implies that, whenever the optimal solution requires $\Omega(n)$ points, our algorithm returns an $(O(1), O(1))$-approximation. As we shall discuss in more detail in the paper, such an approximation is guaranteed for a wide set of instances among which we recall instances containing (at least) a pair of points far (enough) from each other, or such that the unit disk graph induced by $P$ is made up of many connected components. However, we are not able to prove that our algorithm performs well when the points are too close to each other. For this case, in Section 5 we design an $(O(\log n), O(\log n))$-approximation algorithm that, as its first step, builds from $P$ a graph representing all the solutions of our problem up to constant factors for both the number of added points and the diameter value. Then, it applies to such graph the algorithm proposed in [16]. Since the construction of the graph requires polynomial time in the maximum Euclidean distance between any pair of points in $P$, the algorithm runs in polynomial time only for sets of points belonging to a small region in $\mathbb{R}^2$. However, we remark that, when this does not happen, the algorithm described in Section 4 does yield a constant approximation.

## 2. Preliminaries

Let $V$ be a set of points in $\mathbb{R}^2$. The *unit disk graph* induced by $V$, denoted as $\mathtt{udg}(V)$, is the undirected graph over the node set $V$ such that two nodes $u, v \in V$ are adjacent if and only if their Euclidean distance $||uv||$ is at most 1.

We denote by $\mathtt{diam}(V)$ the diameter of $\mathtt{udg}(V)$ and by $\delta(V)$ the maximum Euclidean distance between any pair of points in $V$ rounded up to its closest larger integer:

$$\delta(V) = \lceil \max\{||uv|| : u, v \in P\} \rceil.$$

Needless to say, $\delta(V) \le \mathtt{diam}(V)$.

Let $P, S \subset \mathbb{R}^2$ be such that $P \cap S = \emptyset$; the *P-hop count* between any pair of nodes $u, v \in P$, denoted as $h_{S,P}(u, v)$, is the number of edges in any shortest path between $u$ and $v$ in $\mathtt{udg}(P \cup S)$. The *P-diameter* of $\mathtt{udg}(P \cup S)$, denoted as $\mathtt{diam}(P, S)$, is the maximum $P$-hop count between all pairs of nodes in $P$:

$$\mathtt{diam}(P, S) = \max\{h_{S,P}(u, v) : u, v \in P\}.$$

The *Bounded Diameter with minimum number of Added Nodes* problem (in short MIN-BDAN) is then defined as follows: given a set $P$ of $n$ points in $\mathbb{R}^2$ and $D \in \mathbb{N}$, we ask for a minimum size set $S$ of points in $\mathbb{R}^2$ such that the $P$-diameter of $\mathtt{udg}(P \cup S)$ is at most $D$.

A *feasible solution* for an instance $\langle P, D \rangle$ of MIN-BDAN is any set of points $S \subset \mathbb{R}^2$ such that the $P$-diameter of $\mathtt{udg}(P \cup S)$ is at most $D$. Observe that, if $\delta(P) > D$, no feasible solution exists.

Given a set $P$ of points in $\mathbb{R}^2$, a *geometric Steiner Tree* of $P$ is a tree $T$ connecting all points in $P$ and constituted by line segments; if the intersection point of two line segments in $T$ is not in $P$ it is called a *Steiner point*. The cost of $T$, denoted as $c(T)$, is defined as

$$c(T) = \sum_{(u,v) \in E(T)} ||uv||.$$

A slightly different cost function associated to a geometric Steiner tree $T$ of a set $P \subset \mathbb{R}^2$ can be considered, that is, its *discrete cost* $c_d$ defined as

$$c_d(T) = \sum_{(u,v) \in T} \lceil ||uv|| \rceil.$$

Not surprisingly, if we do not care about the value of the $P$-diameter, there is a strong relation between the size of a feasible solution for an instance $\langle P, D \rangle$ of MIN-BDAN and the discrete cost of a geometric Steiner tree of $P$; this is stated in the following lemma.

**Lemma 1.** *For any $P \subset \mathbb{R}^2$, there exists $S \subset \mathbb{R}^2$ such that $\mathtt{udg}(P \cup S)$ is connected if and only if a geometric Steiner tree $T$ of $P$ exists such that $c_d(T) = |S| + |P| - 1$.*

**Proof.** Let $S$ be a set of points in $\mathbb{R}^2$ such that $\mathtt{udg}(P \cup S)$ is connected and let $T = (P \cup S, E_T)$ be a spanning tree of $\mathtt{udg}(P \cup S)$. Since $T$ is a tree and since $\lceil ||uv|| \rceil = 1$ for any $(u, v) \in E_T$, then

$$c_d(T) = \sum_{(u,v) \in E_T} \lceil ||uv|| \rceil = |P| + |S| - 1.$$

Conversely, let $T$ be a geometric Steiner tree of $P$ and denote as $t$ the number of the Steiner nodes in $T$ and as $E_T$ the set of its edges; from $T$ we compute a set $S$ of nodes such that $\mathtt{udg}(P \cup S)$ is connected. Actually, the set $S$ contains all the Steiner nodes in $T$ and, for any edge $(u, v)$ in $T$, it contains $\lceil ||uv|| \rceil - 1$ points uniformly distributed at distance at most 1 onto the edge $(u, v)$. Hence,

$$\begin{aligned} |S| &= t + \sum_{(u,v) \in E_T} (\lceil ||uv|| \rceil - 1) \\ &= t - (|P| + t - 1) + \sum_{(u,v) \in E_T} \lceil ||uv|| \rceil \\ &= 1 - |P| + c_d(T). \qquad \square \end{aligned}$$

## 3. Hardness

In this section we prove that the decision problem BDAN corresponding to MIN-BDAN is **NP**-hard. An instance of BDAN consists of a set of points $P \subset \mathbb{R}^2$, and of two integer values $k$ and $D$, and asks whether a set $S$ of (at most) $k$ points exists such that the $P$-diameter of $\mathtt{udg}(P \cup S)$ is at most $D$.