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An improved approximation algorithm for the traveling salesman problem with relaxed triangle inequality

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ABSTRACT

Given a complete edge-weighted graph G, we present a polynomial time algorithm to compute a degree-four-bounded spanning Eulerian subgraph of $2G$ that has at most 1.5 times the weight of an optimal TSP solution of G. Based on this algorithm and a novel use of orientations in graphs, we obtain a $(3\beta/4+3\beta^2/4)$-approximation algorithm for TSP with β-relaxed triangle inequality (β-TSP), where $\beta \geq 1$. A graph G is an instance of β-TSP, if it is a complete graph with edge weights $c\colon E(G) \to \mathbb{Q}_{\geq 0}$ that are restricted as follows. For each triple of vertices $u, v, w \in V(G)$, $c(\{u, v\}) \leq \beta(c(\{u, w\}) + c(\{w, v\}))$.

1. Introduction

In the traveling salesman problem we are given a complete edge-weighted graph and we have to find a minimum-weight Hamiltonian tour, i.e., a tour that visits each vertex exactly once. This classical problem has been studied extensively and in many variations. Most of the variations concern restrictions of the weight function. One of the most natural restrictions is to assume that the weight function is a metric. Intuitively this means that we are allowed to take the shortest path to the next vertex to visit, even if this means to visit some of the vertices more than once.

Despite intensive research for more than 30 years, Christofides' algorithm is still the best known approximation algorithm for the metric traveling salesman problem and its approximation ratio is 1.5 [1]. For similar settings, however, a recent fast development has started. For graphic metrics—metrics obtained by taking the lengths of the shortest paths in an unweighted graph as weights—a sequence of improvements was published within a short time frame, with the currently best approximation algorithm of Sebő and Vygen which achieves an approximation ratio of 1.4 [2–5].

Instead of restricting the metric, in this paper we consider a relaxation. We use a relaxation parameter $\beta \geq 1$ and require that the input instance is a complete graph G where the non-negative weight function $c\colon E(G) \to \mathbb{Q}_{\geq 0}$ satisfies the relaxed triangle inequality $c(\{u, w\}) \leq \beta(c(\{u, v\}) + c(\{v, w\}))$ for any three vertices $u, v, w \in V(G)$.

For approximation algorithms, the relaxed triangle inequality was introduced by Bandelt, Crama, and Spieksma [6]. This type of parameterization also provides a suitable type of relaxation in the context of stability of approximation [7] and our result fits into this framework.

The β-relaxed version of metric TSP (β-TSP) was first considered by Andreae and Bandelt [8] who presented a $1.5\beta^2 + 0.5\beta$ approximation algorithm. Subsequently Andreae improved the result to $\beta + \beta^2$ [9]. The next development was due to Böckenhauer et al. [7]. They obtained a $1.5\beta^2$-approximation algorithm, which is better

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than the previous algorithms for $1 < \beta < 2$. Bender and Chekuri [10] independently obtained a 4β approximation algorithm, which is better than the algorithm of Andreae for $\beta > 3$.

1.1. Results and overview of techniques

We provide an improved approximation algorithm for β-TSP.

Theorem 1.1. *There is a polynomial time* $(3\beta/4 + 3\beta^2/4)$*-approximation algorithm for* β*-TSP.*

The approximation ratio of our algorithm outperforms the ratios provided by Andreae [9] and by Böckenhauer et al. [7] for all values of β. The 4β approximation algorithm of Bender and Chekuri [10] still is better than our algorithm for $\beta > 13/3 \approx 4.33$. To obtain our result we first use the matroid version by Király et al. [11] of a bounded degree spanning tree result of Singh and Lau [12], combined with special b-matchings that respect parities of vertex degrees. We obtain a degree-4-bounded spanning Eulerian subgraph in $2G$ for any complete graph G with edge weights $c: E(G) \to \mathbb{Q}$ such that the weight of the computed graph is at most 1.5 times the weight of an optimal TSP solution. Finally, we introduce an orientation technique that provides a cactus graph with useful properties such that the weight of shortcuts within the graph is restricted when constructing the TSP solution. One key insight is that we obtain two disjoint sets of edges such that we have to consider the factor β^2 only for the smaller of the two sets.

2. Preliminaries

All graphs in this paper are allowed to have multiple edges. For convenience of notation, however, we do not distinguish between multi-sets and sets. We handle multiple edges of a graph as separate edges. This way there may be cycles of length two. (We define a cycle to be a simple cycle, i.e., vertices may not be visited twice.)

Given a graph G, $V(G)$ and $E(G)$ are its set of vertices and its set of edges. For a set of edges $F \subseteq E(G)$, we write $c(F)$ as shorthand for $\sum_{e \in F} c(e)$. Similarly, for a graph G we write $c(G)$ as shorthand for $c(E(G))$. A *block* of a graph G is a maximal two-vertex-connected subgraph.

A b-matching of a graph G is a subgraph G' of G with possible additional multiplicities of edges where each vertex has a degree of at most b. We identify a b-matching with its characteristic vector $\mathbf{x}$ of edges, that is, for each edge $e \in E(G)$, $\mathbf{x}$ has an entry $x_e \in \mathbb{N}_0$ and $x_e \geq 1$ if and only if $e \in E(G')$.

Let $G = (V, E)$ be an undirected graph and let M be the $V \times E$ incidence matrix of G. Let $\mathbf{l} \leq \mathbf{m}$ and $\mathbf{a} \leq \mathbf{b}$ be integer vectors in $\mathbb{N}_0^E$ resp. $\mathbb{N}_0^V$ and let S^{odd} and S^{even} be disjoint subsets of V.

Consider the following constraints that impose restrictions on the characteristic vector of the b-matching $\mathbf{x} \in \mathbb{N}_0^E$.

$$\begin{array}{llll} \text{(i)} & \mathbf{l} \leq \mathbf{x} \leq \mathbf{m} & \text{(iii)} & (M\mathbf{x})_v \text{ is odd} \quad \text{if } v \in S^{\text{odd}} \\ \text{(ii)} & \mathbf{a} \leq M\mathbf{x} \leq \mathbf{b} & \text{(iv)} & (M\mathbf{x})_v \text{ is even} \quad \text{if } v \in S^{\text{even}} \end{array} \qquad (1)$$

These constraints specify (i) bounds on the multiplicity of edges in G, (ii) bounds on the degrees of the vertices in G, and (iii, iv) the parities of degrees for specific vertices (since $M\mathbf{x}$ is the vector of vertex degrees).

We slightly abuse notation and, whenever the meaning is clear from the context, we associate an integer $\mathbf{i}$ with the corresponding vector $(i, i, \ldots, i)$. Note that for an integer b', (1) specifies a b'-matching if we set $\mathbf{l} = \mathbf{0}$, $\mathbf{m} = \infty$, $\mathbf{a} = \mathbf{0}$, $\mathbf{b} = \mathbf{b}'$, and $S^{\text{odd}} = S^{\text{even}} = \emptyset$. For our results we need specific b-matchings that we will specify by giving values to the parameters of (1).

The following theorem is Theorem 36.5 in Schrijver's book [13]. (Here, we use a simplified setting. In the original theorem a more general class of graphs can be used.)

Theorem 2.1. *(See Edmonds, Johnson [14].) For any* $\mathbf{c} \in \mathbb{Q}^E$*, an integer vector* $\mathbf{x}$ *maximizing* $\mathbf{c}^T\mathbf{x}$ *over (1) can be found in strongly polynomial time (if it exists).*

Given a graph G, a 1-tree is a subgraph of G composed of a spanning tree on $V(G) \setminus \{v_1\}$ for some $v_1 \in V(G)$ and two edges incident to v_1. If G is a complete graph, then for each choice of v_1 there is a matroid M such that the 1-trees of G are the bases of M [15]. We are interested in 1-trees where additionally the vertices have degree restrictions.

For a vector $\mathbf{b}$ of vertex degrees, a 1-tree T of G is degree $\mathbf{b}$ bounded if the degree of each vertex $v \in V(T)$ is at most $\mathbf{b}_v$. The following theorem follows directly from Király et al. [11] who showed a more general result for matroids.[1] (We run their algorithm for each choice of v_1.)

Theorem 2.2. *Given a complete graph* G *with edge weights* $c: E(G) \to \mathbb{Q}$ *and a vector* $\mathbf{b}$ *of upper bounds on the vertex degrees, there is polynomial time algorithm that computes a 1-tree* T *in* G *such that (a)* T *is degree* $(\mathbf{b}+\mathbf{1})$ *bounded and (b)* $c(T) \leq c(T')$ *for all degree* $\mathbf{b}$ *bounded 1-trees* T' *in* G*.*

For our algorithm we need cactus graphs. Here we use the following definition of cacti which strictly speaking specifies the subclass of 2-edge-connected cacti.

Definition 2.3 *(Cactus).* A graph G is a *cactus* if it is 2-edge connected and all blocks are cycles, i.e., no two cycles share an edge.

Given a graph G, $2G$ is the graph where each edge of G is doubled. A graph G' is a *spanning Eulerian subgraph of* $2G$ if $V(G) = V(G')$, G' is a connected subgraph of $2G$, and each vertex of G' has even degree.

3. The algorithm ALG_β

Before we proceed to the main result, we show an intermediate observation. With the preparation in the pre-

[1] Note that by guessing two consecutive edges and applying straightforward graph modifications, we could also use the predecessor result on bounded degree spanning trees [12].