Information Processing Letters 115 (2015) 877–881

Contents lists available at ScienceDirect

Information Processing Letters

www.elsevier.com/locate/ipl

10-Gabriel graphs are Hamiltonian

Tomáš Kaiser [a,1], Maria Saumell [b,*,2], Nico Van Cleemput [b,2]

[a] Department of Mathematics, Institute for Theoretical Computer Science (CE-ITI), and European Centre of Excellence NTIS (New Technologies for the Information Society), University of West Bohemia, Pilsen, Czech Republic

[b] Department of Mathematics and European Centre of Excellence NTIS (New Technologies for the Information Society), University of West Bohemia, Pilsen, Czech Republic

ARTICLE INFO

Article history:
Received 21 January 2015
Received in revised form 25 May 2015
Accepted 26 May 2015
Available online 1 June 2015
Communicated by R. Uehara

Keywords:
Computational geometry
Proximity graphs
Gabriel graphs
Hamiltonian cycles

ABSTRACT

Given a set S of points in the plane, the *k-Gabriel graph* of S is the geometric graph with vertex set S, where $p_i, p_j \in S$ are connected by an edge if and only if the closed disk having segment $\overline{p_i p_j}$ as diameter contains at most k points of $S \setminus \{p_i, p_j\}$. We consider the following question: What is the minimum value of k such that the k-Gabriel graph of every point set S contains a Hamiltonian cycle? For this value, we give an upper bound of 10 and a lower bound of 2. The best previously known values were 15 and 1, respectively.

© 2015 Elsevier B.V. All rights reserved.

1. Introduction

Let S be a set of n distinct points in the plane. Loosely speaking, a *proximity graph* on S is a graph that attempts to capture the relations of proximity among the points in S. Usually, one defines a reasonable criterion for two points to be considered close to each other, and then the pairs of points that satisfy the criterion are connected in the graph. The study of proximity graphs has been a popular topic in computational geometry, since these graphs not only satisfy interesting theoretical properties, but also have applications in several fields, such as shape analysis, geographic information systems, data mining, computer graphics, or graph drawing (see, for example, [3,17]).

The Delaunay graph and its relatives constitute a prominent family of proximity graphs. In the *Delaunay graph* of S, denoted by $\mathsf{DG}(S)$, $p_i, p_j \in S$ are connected by an edge if and only if there exists a closed disk with p_i, p_j on its boundary that does not contain any point of $S \setminus \{p_i, p_j\}$ (see [11]). It is well known that if S does not contain three collinear or four cocircular points, then $\mathsf{DG}(S)$ is a triangulation of S.

Two related proximity graphs are the relative neighborhood graph and the Gabriel graph. In the *relative neighborhood graph* of S, denoted by $\mathsf{RNG}(S)$, $p_i, p_j \in S$ are connected by an edge if and only if there does not exist any $p_\ell \in S$ such that $d(p_i, p_\ell) < d(p_i, p_j)$ and $d(p_j, p_\ell) < d(p_i, p_j)$, where $d(p,q)$ denotes the Euclidean distance between p and q (see [21]).

Given two points $p_i, p_j \in S$, we denote the closed disk having segment $\overline{p_i p_j}$ as diameter by $\text{C-DISC}(p_i, p_j)$. The *Gabriel graph* of S is the graph in which $p_i, p_j \in S$ are connected by an edge if and only if $\text{C-DISC}(p_i, p_j) \cap S = \{p_i, p_j\}$ (see [16]). We denote the Gabriel graph of S by $\mathsf{GG}(S)$. Notice that $\mathsf{RNG}(S) \subseteq \mathsf{GG}(S) \subseteq \mathsf{DG}(S)$ holds for any point set S.

* Corresponding author.
E-mail addresses: kaisert@kma.zcu.cz (T. Kaiser), saumell@kma.zcu.cz (M. Saumell), cleemput@kma.zcu.cz (N. Van Cleemput).

[1] Supported by project 14-19503S of the Czech Science Foundation.
[2] Supported by the project NEXLIZ CZ.1.07/2.3.00/30.0038, which is co-financed by the European Social Fund and the state budget of the Czech Republic.

http://dx.doi.org/10.1016/j.ipl.2015.05.013
0020-0190/© 2015 Elsevier B.V. All rights reserved.

All of the above graphs are plane, that is, if edges are drawn as line segments, then the resulting drawing contains no crossings. In the last decades, a number of works have been devoted to investigate whether they fulfill other desirable graph-theoretic, geometric, or computational properties. For example, it has been studied whether the vertices of these graphs have bounded maximum or expected degree [19,12,7], whether these graphs are constant spanners [6,14], or whether they are compatible with simple online routing algorithms [18].

A problem that attracted much attention is the Hamiltonicity of Delaunay graphs: Does DG(S) contain a Hamiltonian cycle for every point set S? Dillencourt [13] answered this question negatively by providing an example of a set of points whose Delaunay graph is a non-Hamiltonian triangulation. This naturally raises the question whether there exist variants of the Delaunay graph that do always contain a Hamiltonian cycle.

This problem has been studied for the Delaunay graph in the L_∞ metric. This graph contains an edge between $p_i, p_j \in S$ if and only if there exists an axis-aligned square containing p_i, p_j and no other point in S. Even though Delaunay graphs in the L_∞ metric need not contain a Hamiltonian cycle, they satisfy the slightly weaker property of containing a Hamiltonian path, as shown by Ábrego et al. [2].

Another natural variant of Delaunay graphs which has received some interest is that of *k-Delaunay graphs*, k-DG(S) for short [1]. In this case, the definition is relaxed in the following way: $p_i, p_j \in S$ are connected by an edge if and only if there exists a closed disk with p_i, p_j on its boundary that contains at most k points of $S \setminus \{p_i, p_j\}$. Analogous generalizations lead to *k-Gabriel graphs* and *k-relative neighborhood graphs*. The k-Gabriel graph of S, denoted by k-GG(S), is the graph in which p_i, p_j are connected by an edge if and only if $|\text{C-DISC}(p_i, p_j) \cap S| \le k+2$ (see [20]). The k-relative neighborhood graph of S, denoted by k-RNG(S), is the graph in which p_i, p_j are connected by an edge if and only if there exist at most k points $p_\ell \in S$ such that $d(p_i, p_\ell) < d(p_i, p_j)$ and $d(p_j, p_\ell) < d(p_i, p_j)$ (see [10]).

Notice that 0-DG(S) = DG(S) and, for any $k \ge 0$, k-DG(S) $\subseteq$ $(k+1)$-DG(S). Since k-DG(S) is the complete graph for $k \ge n/2$ [1] and the complete graph is Hamiltonian, the following question arises: What is the minimum value of k such that k-DG(S) is Hamiltonian for every S? Abellanas et al. [1] conjectured that this value is 1, that is, 1-DG(S) is already Hamiltonian. The same question can be formulated for k-GG(S) and k-RNG(S).

The first upper bound for such minimum value of k was given by Chang et al. [9], who proved that 19-RNG(S) is always Hamiltonian.[3] Since, for any $k \ge 0$, k-RNG(S) $\subseteq$ k-GG(S) $\subseteq$ k-DG(S), the result implies that 19-GG(S) and 19-DG(S) are also Hamiltonian. Later, Abellanas et al. [1] improved the bound for the latter graphs by showing that 15-GG(S) (and thus 15-DG(S)) is already Hamiltonian.[4] In this short paper we improve their bound as follows:

Theorem 1. *For any set of points S, the graph* 10-GG(S) *is Hamiltonian.*

We note that related properties of k-Gabriel graphs have been recently considered by Biniaz et al. [4]. In particular, the authors show that 10-GG(S) always contains a Euclidean bottleneck perfect matching, that is, a perfect matching that minimizes the length of the longest edge. Our proof of Theorem 1 actually shows that 10-GG(S) always contains a Euclidean bottleneck Hamiltonian cycle, which is a Hamiltonian cycle minimizing the length of the longest edge. Even though the two results are closely related, there is no direct implication between them.

We prove Theorem 1 in Section 2. Our proof uses the same general strategy as the ones in [1,9]: We select a particular Hamiltonian cycle of the complete graph on S, and we find a value of k such that k-DG(S) contains this Hamiltonian cycle. In Section 3, we show that the best result that can possibly be proved with this particular approach is the Hamiltonicity of 6-Gabriel graphs (we also indicate the existence of an unpublished example [5] showing that the method cannot go beyond 8-GG). We further point out that it might be possible to decrease the value 10 by using a quadratic solver. Finally, we provide an example showing that 1-Gabriel graphs are not always Hamiltonian.

2. Proof of Theorem 1

The first steps of our proof go along the same lines as the arguments in [1] showing that 15-Gabriel graphs are Hamiltonian. The same general strategy was first used in [9]. We provide the details for completeness.

We denote by $\mathcal{H}$ the set of all Hamiltonian cycles of the complete graph on S. Given a cycle $h \in \mathcal{H}$, we define the *distance sequence* of h, denoted $ds(h)$, as the sequence containing the lengths of the edges of h sorted in decreasing order (the *length* of an edge is the length of the straight-line segment connecting its endpoints). Then, we define a strict order on the elements of $\mathcal{H}$ as follows: for $h_1, h_2 \in \mathcal{H}$, we say that $h_1 \succ h_2$ if and only if $ds(h_1) > ds(h_2)$ in the lexicographical order.

Let m be a minimal element of $\mathcal{H}$ with respect to the order that we have just defined. In the remainder of this section we show that all edges of m belong to 10-GG(S), which in particular implies that 10-GG(S) is Hamiltonian.

Let $e = xy$ be any edge of m. We are going to show that e is in 10-GG(S). Without loss of generality, we suppose that $x = (-1, 0)$ and $y = (1, 0)$. For any point p in $\mathbb{R}^2$, we write $\|p\|$ for the distance of p from the origin $o = (0, 0)$.

Let $U = \{u_1, u_2, \ldots, u_\kappa\}$ be the set of points in S different from x, y that are contained in C-DISC(x, y). We want to prove that $\kappa \le 10$. Suppose that, if we traverse the entire cycle m starting from the "directed" edge $\vec{xy}$ and finishing at x, we encounter the vertices of U in the order

[3] Chang et al. [9] define k-RNG(S) in a slightly different way, so k-RNG(S) in their paper is equivalent to $(k-1)$-RNG(S) in our paper.

[4] There also exists an unpublished upper bound of 13 [8].

Download English Version:

https://daneshyari.com/en/article/428850

Download Persian Version:

https://daneshyari.com/article/428850

Daneshyari.com