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Characterization of repetitions in Sturmian words: A new proof

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ABSTRACT

We present a new, dynamical way to study powers (that is, repetitions) in Sturmian words based on results from Diophantine approximation theory. As a result, we provide an alternative and shorter proof of a result by Damanik and Lenz characterizing powers in Sturmian words [6]. Further, as a consequence, we obtain a previously known formula for the fractional index of a Sturmian word based on the continued fraction expansion of its slope.

1. Introduction

In 2003 Damanik and Lenz [6] completely described factors of length n of a Sturmian word which occur as pth powers for every $n \geq 0$ and $p \geq 1$. Damanik and Lenz prove a series of results concerning how factors of a Sturmian word align to the corresponding (finite) standard words. By a careful analysis of the alignment, they obtain the complete description of powers thanks to known results on powers of standard words. Our method is based on the dynamical view of Sturmian words as codings of irrational rotations. Translating word-combinatorial concepts into corresponding dynamical concepts allows us to apply powerful results from Diophantine approximation theory (such as the Three Distance Theorem) providing a more geometric proof of the result of Damanik and Lenz. Our methods allow us to avoid tricky alignment arguments making the proof in our opinion easier to follow. Furthermore, the results allow us to infer a formula for the fractional index of a Sturmian word based on the continued fraction expansion of its slope. This formula and its proof appeared in an earlier paper by Damanik and Lenz [5] and was also established purely combinatorially using alignment arguments. The formula was independently obtained with different methods by Carpi and de Luca [3] and Justin and Pirillo [7]. For partial results and works related to powers in Sturmian words see e.g. the papers of Mignosi [11], Berstel [2], Vandeth [13], and Justin and Pirillo [7].

The paper is organized as follows: in Section 2 we briefly recall results concerning continued fractions and rational approximations and prove the purely number-theoretic and important Proposition 2.2 for later use in Section 4. In Section 3 we state needed facts about Sturmian words with appropriate references. Section 4 contains the main results and their proofs.

2. Continued fractions and rational approximations

Every irrational real number α has a unique infinite continued fraction expansion

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$$\alpha = [a_0; a_1, a_2, a_3, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} \tag{1}$$

with $a_0 \in \mathbb{Z}$ and $a_k \in \mathbb{N}$ for all $k \geq 1$. The numbers a_i are called the *partial quotients* of α. Good references on continued fractions are the books of Khinchin [8] and Cassels [4]. We focus here only on irrational numbers, but we note that with small tweaks much of what follows also holds for rational numbers, which have finite continued fraction expansions.

The *convergents* $c_k = \frac{p_k}{q_k}$ of α are defined by the recurrences

$$\begin{aligned} &p_0 = a_0, && p_1 = a_1 a_0 + 1, && p_k = a_k p_{k-1} + p_{k-2}, && k \geq 2, \\ &q_0 = 1, && q_1 = a_1, && q_k = a_k q_{k-1} + q_{k-2}, && k \geq 2. \end{aligned}$$

The sequence $(c_k)_{k \geq 0}$ converges to α. Moreover, the even convergents are less than α and form an increasing sequence and, on the other hand, the odd convergents are greater than α and form a decreasing sequence.

If $k \geq 2$ and $a_k > 1$, then between the convergents c_{k-2} and c_k there are *semiconvergents* (called intermediate fractions in Khinchin's book [8]) which are of the form

$$\frac{p_{k,l}}{q_{k,l}} = \frac{l p_{k-1} + p_{k-2}}{l q_{k-1} + q_{k-2}}$$

with $1 \leq l < a_k$. When the semiconvergents (if any) between c_{k-2} and c_k are ordered by the size of their denominators, the obtained sequence is increasing if k is even and decreasing if k is odd.

Note that we make a clear distinction between convergents and semiconvergents, i.e., convergents are not a specific subtype of semiconvergents.

For the rest of this paper we make the convention that α refers to an irrational number with a continued fraction expansion as in (1) having convergents $\frac{p_k}{q_k}$ and semiconvergents $\frac{p_{k,l}}{q_{k,l}}$ as above.

A rational number $\frac{a}{b}$ is a *best approximation* of the real number α if for every fraction $\frac{c}{d}$ such that $\frac{c}{d} \neq \frac{a}{b}$ and $d \leq b$ it holds that

$$|b\alpha - a| < |d\alpha - c|\,.$$

In other words, any other multiple of α with a coefficient at most b is further away from the nearest integer than is $b\alpha$. The next proposition shows that the best approximations of an irrational number are connected to its convergents (for a proof see Theorems 16 and 17 of [8]).

Proposition 2.1. *The best rational approximations of an irrational number are exactly its convergents.*

We identify the unit interval $[0, 1)$ with the unit circle $\mathbb{T}$. Let $\alpha \in (0, 1)$ be irrational. The map

$$R : [0, 1) \to [0, 1),\ x \mapsto \{x + \alpha\},$$

where $\{x\}$ stands for the fractional part of the number x, defines a rotation on $\mathbb{T}$. The circle partitions into the intervals $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$. Points in the same interval of the partition are said to be on the same side of 0, and points in different intervals are said to be on the opposite sides of 0. (We are not interested in the location of the point $\frac{1}{2}$.) The points $\{q_k\alpha\}$ and $\{q_{k-1}\alpha\}$ are always on the opposite sides of 0. The points $\{q_{k,l}\alpha\}$ with $0 < l \leq a_k$ always lie between the points $\{q_{k-2}\alpha\}$ and $\{q_k\alpha\}$; see (3).

We measure the shortest distance to 0 on $\mathbb{T}$ by setting

$$\|x\| = \min\{\{x\}, 1 - \{x\}\}.$$

We have the following facts for $k \geq 2$ and for all l such that $0 < l \leq a_k$:

$$\|q_{k,l}\alpha\| = (-1)^k (q_{k,l}\alpha - p_{k,l}), \tag{2}$$

$$\|q_{k,l}\alpha\| = \|q_{k,l-1}\alpha\| - \|q_{k-1}\alpha\|. \tag{3}$$

We can now interpret Proposition 2.1 as

$$\min_{0<n<q_k} \|n\alpha\| = \|q_{k-1}\alpha\|, \quad \text{for } k \geq 1. \tag{4}$$

Note that rotating preserves distances; a fact we will often use without explicit mention. In particular, the distance between the points $\{n\alpha\}$ and $\{m\alpha\}$ is $\||n - m|\alpha\|$. Thus by (4) the minimum distance between the distinct points $\{n\alpha\}$ and $\{m\alpha\}$ with $0 \leq n, m < q_k$ is at least $\|q_{k-1}\alpha\|$. The formula (4) tells what is the point closest to 0 among the points $\{n\alpha\}$ for $1 \leq n \leq q_k - 1$. We are also interested to know the point closest to 0 on the side opposite to $\{q_{k-1}\alpha\}$. The next result is very important and concerns this.

Proposition 2.2. *Let α be an irrational number. Let n be an integer such that $0 < n < q_{k,l}$ with $k \geq 2$ and $0 < l \leq a_k$. If $\|n\alpha\| < \|q_{k,l-1}\alpha\|$, then $n = mq_{k-1}$ for some integer m such that $1 \leq m \leq \min\{l, a_k - l + 1\}$.*

Proof. Suppose that $\|n\alpha\| < \|q_{k,l-1}\alpha\|$, and assume for a contradiction that the point $\{n\alpha\}$ is on the same side of 0 as $\{q_{k-2}\alpha\}$. Since $n < q_{k,l}$, we conclude that $n \neq q_{k,r}$ for $r \geq l$. By (3) and our assumption that $\|n\alpha\| < \|q_{k,l-1}\|$, we see that $n \neq q_{k,r}$ with $0 \leq r \leq l-1$. As $\|n\alpha\| > \|q_k\alpha\|$ by (4), we infer that the point $\{n\alpha\}$ must lie between the points $\{q_{k,l'}\alpha\}$ and $\{q_{k,l'+1}\alpha\}$ for some l' such that $0 \leq l' < a_k$. The distance between the points $\{n\alpha\}$ and $\{q_{k,l'}\}$ is less than $\|q_{k-1}\alpha\|$. By (4), it must be that $q_{k,l'} \geq q_k$; a contradiction.

Suppose for a contradiction that n is not a multiple of q_{k-1}. Then the point $\{n\alpha\}$ lies between the points $\{tq_{k-1}\alpha\}$ and $\{(t+1)q_{k-1}\alpha\}$ for some t such that $0 < t < \lfloor 1/\|q_{k-1}\alpha\| \rfloor$. As $\{n\alpha\}$ is on the same side of 0 as the point $\{q_{k-1}\alpha\}$, it follows that $\|n\alpha\| > \|tq_{k-1}\alpha\|$ and $\|tq_{k-1}\alpha\| = t\|q_{k-1}\alpha\|$. The distance between the points $\{n\alpha\}$ and $\{tq_{k-1}\alpha\}$ is less than $\|q_{k-1}\alpha\|$, so by (4) it must be that $tq_{k-1} \geq q_k = a_k q_{k-1} + q_{k-2}$. Thus necessarily $t > a_k$. Using (3) we see that the distance between the points $\{q_k\alpha\}$ and $\{q_{k-2}\alpha\}$ is $a_k\|q_{k-1}\alpha\|$. Since $\|q_k\alpha\| < \|q_{k-1}\alpha\|$, we infer that

$$\begin{aligned} \|q_{k,l-1}\alpha\| \leq \|q_{k-2}\alpha\| &= a_k\|q_{k-1}\alpha\| + \|q_k\alpha\| \\ &< (a_k + 1)\|q_{k-1}\alpha\|. \end{aligned} \tag{5}$$

Therefore by our assumption,