Information Processing Letters 115 (2015) 521–526

Contents lists available at ScienceDirect

Information Processing Letters

www.elsevier.com/locate/ipl

Graphs with six distinct matching roots ☆

Hailiang Zhang [a], Guanglong Yu [b], Shanlin Li [a]

[a] Department of Mathematics, Taizhou University, Linhai, 317000, Zhejiang, China
[b] Department of Mathematics, Yancheng Teachers University, Yancheng, 224002, Jiangsu, China

ARTICLE INFO

Article history:
Received 26 February 2014
Received in revised form 2 October 2014
Accepted 13 October 2014
Available online 22 October 2014
Communicated by Jinhui Xu

Keywords:
Graph algorithms
Matching polynomial
Matching equivalent
Matching unique

ABSTRACT

We investigate graphs whose matching polynomial has six distinct roots. We show that for those graphs if its order $n > 6$ then they are determined by their matching polynomials.

1. Introduction

Let $G(V, E)$ be a *simple graph*, $V(G), E(G)$ be its vertex set and edge set, respectively. Let M be a matching of G and $m(G, k)$ be the number of ways in which k independent edges can be selected in $E(G)$. In [2] [CH. 4], Cvetković, Doob, Gutman and Torgašev denote the *matching polynomial* of G with n vertices and m edges to be

$$M_G(x) = \sum_{k \geq 0}^{n/2} (-1)^k m(G, k) x^{n-2k}. \quad (1)$$

Two nonisomorphic graphs with the same matching polynomials are said to be *comatching*, write as $G \sim H$. A graph G is said to be *matching unique* if it has no comatching graph. We define *the root system* of G to be the multiset of the roots of the matching polynomial of G. We denote it by

$$R(G) = \{\theta_1^{m_1}, \theta_2^{m_2}, \dots, \theta_n^{m_n}\}, \quad (2)$$

where m_i is the multiplicity of θ_i as a root of $M_G(x)$.

So far, only few classes of matching unique graphs are known. For example Beezer and Farrell in [1] have shown that 2-regular graphs, cages, $mK_{r,r}$, Moore graphs with given degree and odd girth are matching unique. The determination of graphs with few distinct roots of its matching polynomial is studied by Ebrahim. In [4], he determines graphs with at most five distinct matching roots and shows that the *Friendship graph* F_n is matching unique.

In this work we determine all graphs with six distinct matching roots. As a result, we find a new class of matching unique graph. In particular, we give matching polynomials and matching roots of all graphs of order 6 and give their comatching graphs (see Appendix [6]).

2. Preliminaries

Let P_n be a path on n vertices, $K_{1,s}$ be a star on $s+1$ vertices and K_n be a complete graph on n vertices. Obviously $P_1 \cong K_1$, $P_2 \cong K_2$ and $P_3 \cong K_{1,2}$.

Lemma 2.1. *(See [2].) Let G be a graph with $u \in V(G)$, and suppose the neighborhood of u is $\Gamma(u) = \{v_1, v_2, \dots, v_d\}$. Then*

$$M_G(x) = x M_{G\backslash u}(x) - \sum_{i=1}^{d} M_{G\backslash uv_i}(x). \quad (3)$$

☆ Supported by NSFC (No. 11271315), ZJNF (No. Y6110054).

E-mail addresses: rockzhang76@tzc.edu.cn (H. Zhang), yglong01@163.com (G. Yu), tztylsl@126.com (S. Li).

http://dx.doi.org/10.1016/j.ipl.2014.10.012
0020-0190/

If $G_1, G_2, \cdots, G_k$ are the components of G, then

$$M_G(x) = \prod_{i=1}^{k} M_{G_i}(x). \tag{4}$$

Roots of a matching polynomial of any graph are real numbers. Moreover it has the *interlacing* property (see [5]).

Lemma 2.2. *(See [5].) Let G be a graph and u be a vertex of it. Then the roots of $M_{G\backslash u}(x)$ interlace those of $M_G(x)$, i.e. if $\theta_1 \geq \theta_2 \geq \ldots \geq \theta_n$ and $\eta_1 \geq \eta_2 \geq \ldots \geq \eta_{n-1}$ are the matching roots of G and $G-u$, respectively, then*

$$\theta_1 \geq \eta_1 \geq \theta_2 \geq \eta_2 \geq \ldots \geq \eta_{n-1} \geq \theta_n.$$

Lemma 2.3. *(See [3].) For a connected graph G, if $mult(\theta, G) \geq 2$, then there is a vertex u of G such that $mult(\theta, G-u) \geq mult(\theta, G)$.*

By following Ghorbani's definition $S(r,k,t,p,q)$ are graphs obtained by adding a new vertex u to the graph $rK_{1,k} \cup tK_1$ and joining it to the other vertices by $p+q$ edges such that the resulting graph is connected, and u is adjacent with q centers of the stars (for $K_{1,1}$ either of the vertices may be consider as center). Clearly $r+t \leq p+q \leq r(k+1)+t$ and $0 \leq q \leq r$.

Lemma 2.4. *(See [4].) Let G be a connected graph, and let $z(G)$ be the number of distinct matching roots. Then*

1. *if $z(G)=2$, then $G \cong K_2$;*
2. *if $z(G)=3$, then G is either a star or K_3;*
3. *if $z(G)=4$, then G is a non-star graph with 4 vertices;*
4. *if $z(G)=5$, then G is one of the graphs $S(1,k,0,s,0)$, $S(1,k,0,s,1)$, $S(1,1,t,t+1,0)$, $S(1,1,t,t+1,1)$, $S(r,1,0,p,q)$,*

for some integers k,s,t,p,q, or a connected non-star graph with 5 vertices.

For convenience, let $G^{(i)}$ denote the connected graph which has i distinct matching roots. It is obvious that $G^{(1)} = \{nK_1\}$, and $R(G^{(1)}) = \{0^n\}$.

Remark 2.5. It is well known that roots of the cubic polynomial ax^3+bx^2+cx+d are given in terms of the coefficients and

$$\Delta = 18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2$$

is called discriminant

1. For the case $\Delta > 0$, the polynomial has three distinct real roots;
2. For the case $\Delta = 0$ the polynomial has a multiple root and all its roots are real;
3. For the case $\Delta < 0$ the polynomial has one real root and two nonreal complex roots.
4. If the cubic polynomial is a factor of a matching polynomial of a simple graph, then $b=0$ and only has real roots.

For a polynomial to the 4th power and 6th power we let $y = x^2$ and decrease it to a quadric or cubic polynomials.

It is well known that the multiplicity of 0 as a root of the matching polynomial is equal to the number of vertices which is unsaturated by the maximal matching of G. Hence we easily have the following Remark 2.6.

Remark 2.6. If a graph has six distinct matching roots, then zero is not a root of the matching polynomial of a graph, furthermore the graph has a perfect matching.

3. All graphs with six distinct matching roots

We define a family of graphs which will be used frequently in this paper. Let $\mathcal{S}(H_1, H_2, H_3) = \{S(rH_1, sH_2, tH_3, m, p, q) : r, s, t, m, p, q \in Z^+ \cup \{0\}\}$ be a set of graphs which are obtained by adding a new vertex u to $rH_1 \cup sH_2 \cup tH_3$, and join to the m vertices of H_1, p vertices of H_2 and q vertices of H_3 with proper m, p and q to make the resulting graph $S(rH_1, sH_2, tH_3, m, p, q)$ connected, where H_i are called *generate subgraphs* of G.

Theorem 3.1. *The following graphs do not have precisely six distinct matching roots:*

$$\mathcal{S}(K_3, K_2) = \left\{S(K_3, rK_2, i, s)\right\},$$
$$\mathcal{S}(K_{1,2}, K_2) = \left\{S(K_{1,2}, rK_2, i, s)\right\},$$

and

$$\mathcal{S}(P_5, K_2) = \left\{S(P_5, rK_2, i, s)\right\}, \quad 1 \leq i \leq 5,\ r \leq s \leq 2r.$$

Proof. For simplicity, let H be the graph that is obtained by adding a new vertex u and connecting u to rK_2 by $s(r \leq s \leq 2r)$ edges. Let $g(x) = (x^2-1)^{r-1}$. By Lemma 2.1 the matching polynomials of $H_1, H_2, \ldots, H_8$ (see Fig. 1) are:

$$M_{H_1}(x) = g(x)\left(x^6 - (s+5)x^4 + (3s+5)x^2 - 1\right),$$
$$M_{H_2}(x) = g(x)\left(x^6 - (s+6)x^4 + (3s+7)x^2 - 2\right),$$
$$M_{H_3}(x) = g(x)\left(x^6 - (s+7)x^4 + (3s+9)x^2 - 3\right),$$

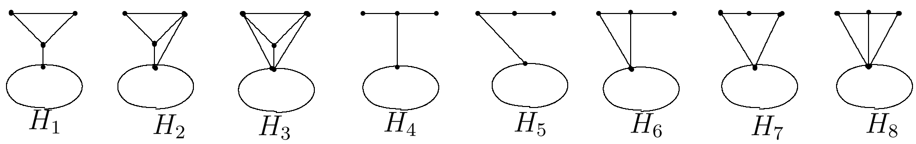

Fig. 1. $S(K_3, rK_2, i, s)$ and $S(K_{1,2}, rK_2, i, s)$, $i = 1, 2, 3$.