Information Processing Letters 115 (2015) 431–434

Contents lists available at ScienceDirect

Information Processing Letters

www.elsevier.com/locate/ipl

On strong centerpoints

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ARTICLE INFO

Article history:
Received 4 November 2013
Received in revised form 8 November 2014
Accepted 11 November 2014
Available online 13 November 2014
Communicated by R. Uehara

Keywords:
Discrete geometry
Centerpoint
Strong centerpoint
Convex polytopes
Hyperplanes

ABSTRACT

Let P be a set of n points in $\mathbb{R}^d$ and $\mathcal{F}$ be a family of geometric objects. We call a point $x \in P$ a strong centerpoint of P w.r.t. $\mathcal{F}$ if x is contained in all $F \in \mathcal{F}$ that contains more than cn points of P, where c is a fixed constant. A strong centerpoint does not exist even when $\mathcal{F}$ is the family of halfspaces in the plane. We prove the existence of strong centerpoints with exact constants for convex polytopes defined by a fixed set of orientations. We also prove the existence of strong centerpoints for abstract set systems with bounded intersection.

1. Introduction

Let P be a set of n points in $\mathbb{R}^d$. A point $x \in \mathbb{R}^d$ is said to be a *centerpoint* of P if any halfspace that contains x contains at least $\frac{n}{d+1}$ points of P. Equivalently, x is a centerpoint if and only if x is contained in every convex object that contains more than $\frac{d}{d+1}n$ points of P. It has been proved that a centerpoint exists for any point set P and the constant $\frac{d}{d+1}$ is tight [17]. We use the second definition of the centerpoint throughout this paper.

The notion of centerpoint has found many applications in statistics, combinatorial geometry, geometric algorithms, etc. [4,13,14,19]. Linear time algorithms to compute approximate centerpoint are given in [6,10–12,18]. Jadhav and Mukhopadhyay [9] gave a linear time algorithm to compute a centerpoint in the plane. Chan [5] gave a randomized algorithm that computes the centerpoint in $\mathbb{R}^d$ in $O(n^{d-1})$ time.

The centerpoint theorem with better constants has been proved for special classes of convex objects. Aronov et al. [1] proved tight bounds for centerpoint for the family of halfplanes, axis-parallel rectangles and disks in $\mathbb{R}^2$. Another well-studied generalization of centerpoint is to allow more than one point. This is related to an area called weak ϵ-nets.

$N \subset P$ is said to be a (strong) ϵ-net of P w.r.t. a family of geometric objects $\mathcal{R}$ if $N \cap R \neq \emptyset$ for all $R \in \mathcal{R}$ that contains more than ϵn points of P. N is called a weak ϵ-net if N is not restricted to be a subset of P but is allowed to be any subset of $\mathbb{R}^d$. Haussler and Welzl [8] showed that small-sized ϵ-nets exist for range spaces of bounded VC-dimension. Small ϵ-net question investigates the bounds on ϵ when the size of ϵ-net is fixed as a small constant [1,3,7,15]. Note that a centerpoint is a weak ϵ-net of size one, w.r.t. convex objects.

In general, a centerpoint need not be a point of P and can be any point in $\mathbb{R}^d$. In this paper, we study the question of enforcing the centerpoint to be a point of P. We call such a centerpoint a strong centerpoint.

We now define strong centerpoints in an abstract setting.

Definition 1. Let P be a set of n elements and $\mathcal{S}$ be a family of subsets of P. Then $p \in P$ is called the *strong centerpoint* of P w.r.t. $\mathcal{S}$ if $p \in S$ for all $S \in \mathcal{S}$ such that $|S| > cn$, where $0 < c < 1$ is a fixed constant.

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http://dx.doi.org/10.1016/j.ipl.2014.11.004

It is easy to see that a strong centerpoint does not exist even when P is a set of n points in $\mathbb{R}^d$ and $\mathcal{S}$ is defined by halfspaces. Let P be a set of n points in convex position. For any point $p \in P$, there exists a halfspace that contains all the points in $P \setminus \{p\}$. Therefore, a strong centerpoint does not exist for halfspaces, and therefore, for disks and convex objects. Ashok et al. [2] proved the existence of strong centerpoints for axis-parallel boxes in $\mathbb{R}^d$. To the best of our knowledge, no other results on strong centerpoints are known.

In this paper, we study the strong centerpoint question and prove tight bounds for some classes of geometric and abstract objects.

1.1. Our results

1. We prove a strong centerpoint exists for a special class of convex polytopes viz. convex polytopes defined by a set of fixed orientations. Let P be a set of n points in $\mathbb{R}^d$ and $\mathcal{F}$ represent the set of convex polytopes defined by a set of k fixed orientations. Then there exists a strong centerpoint $p \in P$ such that p is contained in all $F \in \mathcal{F}$ that contains more than $(1 - \frac{1}{k})n$ points of P. Moreover, this bound is tight. Our proof is constructive and can be converted into a linear time algorithm to compute such a strong centerpoint. Our argument is a generalization of a construction given in Theorem 2 of [2].
2. We prove the existence of a strong centerpoint for set systems with "bounded intersection". Let $(P, \mathcal{S}_k)$ be a set system where P is a set of n elements and $\mathcal{S}_k$ is a collection of subsets of P with the property that the intersection of any k subsets in $\mathcal{S}_k$ is either equal to the intersection of strictly fewer sets among them or contains at most one element of P. We prove that a strong centerpoint $p \in P$ exists such that p is contained in all $S \in \mathcal{S}_k$ such that $|S \cap P| > (1 - \frac{1}{k})n$. Moreover, this bound is tight.

Section 2 gives some definitions and preliminary results that will be used in subsequent sections. Section 3 proves the existence of strong centerpoints and gives tight bounds for the family of convex polytopes defined by a set of fixed orientations. In Section 4, we prove the existence of strong centerpoint for set systems with bounded intersection.

2. Definitions and preliminary results

In this section, we give some definitions and preliminary results that will be used in subsequent sections.

Definition 2. The *orientation* of a halfspace is the direction of the outward normal to that halfspace.

Note that if two halfspaces H_1 and H_2 are of the same orientation then one of them is contained in the other.

Definition 3. Let C be a convex polytope in $\mathbb{R}^d$. Let $H_1, \cdots, H_k$ be the halfspaces defined by faces of C such that $C = \bigcap_{i=1}^k H_i$. We call $H_1, \cdots, H_k$ the defining halfspaces of C.

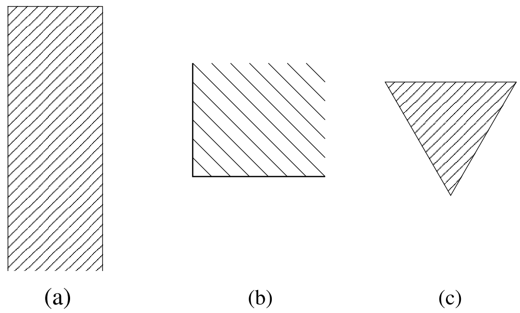

Fig. 1. Examples of Convex Polytopes defined by a fixed set of Orientations: (a) skylines in $\mathbb{R}^2$, (b) orthants of fixed orientations in $\mathbb{R}^2$, (c) downward facing equilateral triangles.

For a general convex polytope C, the defining halfspaces of C can be of any orientation. We consider a class of convex polytopes where the orientations of the defining halfspaces belong to a fixed set.

Let $\mathcal{O}$ be a set of orientations.

Definition 4. A family of convex polytopes $\mathcal{C}$ is said to be *defined by* $\mathcal{O}$ if for any $C \in \mathcal{C}$, the orientations of all the defining halfspaces of C belong to $\mathcal{O}$.

Many common classes of geometric objects fall into this category. For example, axis-parallel boxes are defined by a set of $2d$ fixed orientations viz. direction of positive and negative axes in all the d dimensions. Some other geometric objects that fall into this category are:

- Skylines in $\mathbb{R}^d$: Skylines are axis-parallel boxes where one fixed axis, say the negative axis in the dth dimension, is unbounded. These are convex polytopes defined by $2d - 1$ fixed orientations viz., direction of positive axes in all the d dimensions and direction of negative axes in all d dimensions except the last one. Fig. 1(a) shows skylines in $\mathbb{R}^2$.
- Orthants of fixed orientation in $\mathbb{R}^d$: Orthants in $\mathbb{R}^d$ can be considered as intersection of d mutually orthogonal halfspaces. For orthants of fixed orientation, the orientations of the defining halfspaces are fixed, say as the direction of positive axes in all the d dimensions. Fig. 1(b) shows orthants in $\mathbb{R}^2$.
- Downward facing equilateral triangles: These are equilateral triangles with one side parallel to the X-axis and the corner opposite to this side lying below it [16]. Therefore, this family of triangles are defined by a set of three fixed orientations viz., directions at 90°, 210° and 330° with positive X-axis (see Fig. 1(c)).
- Homothets of a k-faced convex polytope where k is a fixed constant.

Definition 5. A set system $(P, \mathcal{S}_k)$ is said to be a *set system of bounded intersection* if it satisfies the following property: For any k sets in $\mathcal{S}_k$, their intersection is either equal to the intersection of strictly fewer sets among them or contains at most one element of P.