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Approximation algorithms for maximum independent set of a unit disk graph

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ABSTRACT

We propose a 2-approximation algorithm for the maximum independent set problem for a unit disk graph. The time and space complexities are $O(n^3)$ and $O(n^2)$, respectively. For a penny graph, our proposed 2-approximation algorithm works in $O(n \log n)$ time using $O(n)$ space. We also propose a polynomial-time approximation scheme (PTAS) for the maximum independent set problem for a unit disk graph. Given an integer $k > 1$, it produces a solution of size $\frac{1}{(1+\frac{1}{k})^2}|OPT|$ in $O(k^4 n^{\sigma_k \log k} + n \log n)$ time and $O(n + k \log k)$ space, where OPT is the subset of disks in an optimal solution and $\sigma_k \leq \frac{7k}{3} + 2$. For a penny graph, the proposed PTAS produces a solution of size $\frac{1}{(1+\frac{1}{k})}|OPT|$ in $O(2^{2\sigma_k} nk + n \log n)$ time using $O(2^{\sigma_k} + n)$ space.

1. Introduction

Unit disk graphs play an important role in formulating several problems in mobile ad hoc networks. A *unit disk graph* $G = (V, E)$ is the intersection graph of a set of circular disks $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$, placed in $\mathbb{R}^2$, each having diameter 1. The center of disk C_i is denoted by c_i. Each vertex $v_i \in V$ corresponds to a disk C_i, and an edge $(v_i, v_j) \in E$ indicates that the corresponding pair of unit disks C_i and C_j intersect, i.e., $\delta(c_i, c_j) \leq 1$, where $\delta(a, b)$ is the Euclidean distance between a pair of points $a, b \in \mathbb{R}^2$. In a mobile network, if all its base stations have the same range of transmission, then these can be viewed as the vertices of a unit disk graph. Various practical problems on this network can be formulated in terms of a unit disk graph. In this paper, we consider the problem of finding a maximum independent set (MIS) in a given unit disk graph, where the co-ordinates (x_i, y_i) of the center c_i of each disk C_i are given.

The MIS problem for unit disk graph is known to be NP-complete [2]. Thus, research on this topic is concentrated on designing efficient approximation algorithms. Most of the related works assume that the geometric representation, i.e., the layout of the set of unit disks is given. In such an environment, the MIS for unit disk graph is defined as follows:

> *Given a set $\mathcal{C}$ of n circular disks, each of diameter 1, placed arbitrarily in $\mathbb{R}^2$, find a subset OPT of non-intersecting disks in $\mathcal{C}$ such that OPT has the maximum cardinality among all possible subsets of non-intersecting disks in $\mathcal{C}$.*

Previous works: A dynamic programming based shifting strategy was used by Erlebach et al. [7] to design a

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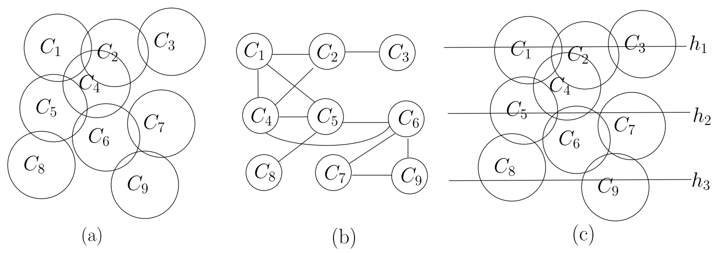

Fig. 1. (a) A layout of unit disks, (b) corresponding unit disk graph, and (c) horizontal lines at distance 1 dividing the region into strips.

polynomial-time approximation scheme (PTAS) for finding a maximum weighted independent set (disjoint disks of maximum total weight) in an intersection graph of n disks of arbitrary radii. They proposed a $(1+\frac{1}{k+1})^2$-approximation algorithm[1] in $n^{O(k^4)}$ time, where k is an integer greater than 1. Matsui [12] showed that if the centers of a set of n unit disks lie inside a region bounded by a pair of parallel lines at a distance at most k apart, then an optimal MIS can be obtained in $O(n^{4\lceil\frac{2k}{\sqrt{3}}\rceil})$ time. He also gave an approximation algorithm for the MIS problem for unit disk graphs, that produces a solution of size $(1-\frac{1}{r})|OPT|$, and runs in $O(rn^{4\lceil\frac{2(r-1)}{\sqrt{3}}\rceil})$ time using $O(n^{2r})$ space, for any positive integer $r \geq 2$. Here OPT is the subset of disks in an optimum solution.

The concept of *thickness* was introduced by van Leeuwen [11] to propose a fixed parameter tractable algorithm for the MIS problem for a unit disk graph. An instance of a unit disk layout is said to have thickness τ if the region containing the disks can be split into a set of strips of width 1 such that each strip contains at most τ disk centers. Further, he showed that an instance of the MIS problem with thickness τ can be solved in $O(\tau^2 2^{2\tau} n)$ time.

Agarwal et al. [1] proposed a 2-approximation algorithm for the MIS problem for the rectangle intersection graph corresponding to a given set of rectangles of fixed size. They also proposed a PTAS for this MIS problem which produces a $(1+\frac{1}{k})$-approximation result in $O(n\log n + n^{2k-1})$ time for any positive integer k. For a set of arbitrary squares or rectangles of bounded aspect ratio in $\mathbb{R}^d$, Chan [4] proposed a PTAS that runs in $O(n^{1/\epsilon^{d-1}})$ time and space with $0 < \epsilon \ll 1$. They improved the space complexity to $O(n)$ by sacrificing the running time to $O(n^{1/\epsilon^d})$. Recently, Chan and Har-Peled [5] addressed the MIS problem for pseudo-disks in the plane. In the unweighted case, for a set of n pseudo-disks, their algorithm produces a solution of size $(1-\varepsilon)|OPT|$ in $O(n^{O(1/\varepsilon^2)})$ time; this result is similar to Erlebach's results [7] for n disks of arbitrary radii. For the weighted case, they proposed an $O(n^3)$ time algorithm to produce an independent set of total weight $\Omega(|OPT|)$, where OPT is a subset of independent disks that produces maximum weight over all possible independent sets, provided the set of pseudo-disks has linear union complexity.

Our main results: First, we propose a 2-approximation algorithm for the MIS problem for a given unit disk graph. The time and space complexities of our proposed algorithm are $O(n^3)$ and $O(n^2)$, respectively. The perspective of this algorithm is that (i) *the best known constant factor approximation algorithm* available for this problem achieves an approximation factor 3 with time complexity $O(n^2)$ [13], and (ii) by plugging $r = 2$ in Matsui's PTAS result for the MIS problem of unit disk graph [12], we get a 2-approximation algorithm that runs in $O(n^8)$ time using $O(n^4)$ space. Hence, our 2-approximation algorithm is much faster than the best known 2-approximation algorithm for the MIS problem for unit disk graphs.

Second, we consider the MIS problem for a *penny graph*, which is a special case of unit disk graph where the corresponding unit disks do not overlap, i.e., the distance between any two disk centers is at least 1. Here an intersection between a pair of disks implies that the distance between corresponding centers is exactly 1. The MIS problem for the penny graph is also NP-complete [3]. Our algorithm produces a 2-approximation result for a penny graph in $O(n\log n)$ time.

Next, we propose a PTAS for the MIS problem for the unit disk graph. It needs to optimally solve a subproblem of the MIS problem where the centers of a set of unit disks lie inside a square region of size $k \times k$. We show that this problem can be solved in $O(k^3 m^{\sigma_k \log k})$ time using $O(k\log k + \min(\varphi_k, m))$ space, where m is the input size of the subproblem, $\sigma_k \leq \frac{7k}{3} + 2$ and $\varphi_k \leq 9(\lceil\frac{k}{2}\rceil)^2$. We then apply the two level shifting strategy of Hochbaum and Maass [10] to get a solution of the original MIS problem, that is of size at least $\frac{1}{(1+\frac{1}{k})^2}|OPT|$, in $O(k^4 n^{\sigma_k \log k})$ time using $O(n + k\log k)$ space, where n is the input size of the original MIS problem. The time complexity of our algorithm is comparable with that of [12]; but its space complexity is $O(n + k\log k)$ compared to $O(n^{2k})$ of [12]. For penny graphs, the time and space complexities of our PTAS are $O(2^{2\sigma_k} nk + n\log n)$ and $O(2^{\sigma_k} + n)$, respectively.

2. MIS problem for unit disk graphs

A layout of a set $\mathcal{C}$ of unit disks and the corresponding unit disk graph are shown in Figs. 1(a) and 1(b), respectively. An independent set in this unit disk graph consists of a set of disks $I \subseteq \mathcal{C}$ which are mutually non-intersecting. The objective of the MIS problem is to find the largest subset of disks in $\mathcal{C}$ which are mutually non-intersecting.

As in the 2-approximation algorithm for the MIS problem of fixed width rectangle intersection graph [1], we

[1] An approximation algorithm for the MIS problem is said to be an α-approximation algorithm if it produces a solution of size $\frac{1}{\alpha}|OPT|$, where OPT is the subset of disks in an optimal solution of the said problem.