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Holographic algorithms on bases of rank 2

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ABSTRACT

An essential problem in the design of holographic algorithms is to decide whether the required signatures can be realized under a suitable basis transformation (SRP). For holographic algorithms with matchgates on domain size 2, [1,2,4,5] have built a systematical theory. In this paper, we reduce SRP on domain size $k \geq 3$ to SRP on domain size 2 for holographic algorithms with matchgates on bases of rank 2. Furthermore, we generalize the collapse theorem of [3] to domain size $k \geq 3$.

1. Introduction

L. Valiant [10] introduced holographic algorithms with matchgates. Computation in these algorithms is expressed and interpreted through a choice of linear basis vectors in an exponential "holographic" mix. Then the actual computation is carried out, via the Holant Theorem, by the Fisher–Kasteleyn–Temperley algorithm for counting the number of perfect matchings in a planar graph. This methodology has produced polynomial time algorithms for a variety of problems. No polynomial time algorithms were known for any of these problems, and some minor variations are known to be NP-hard.

For example, Valiant showed that the restrictive SAT problem $\sharp_7$Pl-Rtw-Mon-3CNF (counting the number of satisfying assignments of a planar read-twice monotone 3CNF formula, modulo 7) is solvable in P [11]. The same counting problem $\sharp$Pl-Rtw-Mon-3CNF without mod 7 is known to be $\sharp$P-complete and the problem mod 2 is $\oplus$P-complete. The surprising tractability mod 7 is due to the unexpected existence of some basis transformations for matchgate signatures.

Following holographic algorithms with matchgates, Cai, Lu and Xia gave holographic algorithms with Fibonacci gates in [6]. To build the dichotomy theorems of counting problems, some other holographic algorithms are introduced. In the important results about the counting problems that are achieved recently, these holographic algorithms played important roles [7–9]. The reason that these algorithms are called holographic algorithms is that they all use holographic reduction between two problems that agree on the output for every input. This shows that holographic algorithms can prove not only tractability, but also hardness. In the following of the present paper, we prove our results in the framework of holographic algorithms with matchgates although there is actually very little dependence on matchgates in our proof (this is pointed out by the referee).

For a general CSP-type counting problem, one can assume there is a natural parameter k, called its domain size. This is the range over which variables take values. For example, Boolean CSP problems all have domain size 2.

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A k-coloring problem on graphs has domain size k. In holographic algorithms with matchgates one considers a linear transformation, which can be expressed as a $2^\ell \times k$ matrix $M = (\alpha_1, \alpha_2, \cdots, \alpha_k)$. This is called a basis of k components, and ℓ is called the size of the basis.[3] A holographic algorithm is said to be on domain size k if the respective signatures are realized by matchgates using a basis of k components. When designing a holographic algorithm for any particular problem, an essential step is to decide whether there is a linear basis for which certain signatures of both generators and recognizers can be simultaneously realized. This is called Simultaneous Realizability Problem (SRP).

For SRP of holographic algorithms with matchgates on domain size 2, a systematic theory has been built in [1,2,4,5]. Recently, Valiant gave polynomial time algorithms for some interesting problems by holographic algorithms on 2×3 bases in [12], i.e. the domain size is 3. To understand the power of holographic algorithms, we need to consider signatures on domain size $k \geq 3$. In the present paper, we give a method to reduce SRP on domain size $k \geq 3$ to SRP on domain size 2 if the signatures are realized on a basis of rank 2.

Obviously, utilizing bases of a higher size is always a theoretic possibility which may allow us to devise more holographic algorithms. But Cai and Lu proved a surprising result for holographic algorithms with matchgates on domain size 2 in [3]: Any holographic algorithms on domain size 2 and a basis of size $\ell \geq 2$ which employs at least one non-degenerate generator can be simulated on a basis of size 1. This is the collapse theorem for holographic algorithms with matchgates on domain size 2. In [6], the collapse theorem for holographic algorithms with Fibonacci gates are given in similar. In this paper, we give a collapse theorem for holographic algorithms with matchgates on a $2^\ell \times k$ basis M, where M has rank 2.

The above results are proved by ruling out a trivial case, which happens when all the recognizers or generators are degenerate. Holographic algorithms which only use degenerate recognizers or generators are trivial [3].

2. Background

In this section, we review some definitions and results. More details can be found in [1,2,4,10,11].

Let $G = (V, E, \omega)$ be a weighted undirected planar graph, where ω assigns edge weights. A generator (resp. recognizer) matchgate Γ is a tuple (G, X) where $X \subseteq V$ is a set of external output (resp. input) nodes. The external nodes are ordered counterclockwise on the external face.

For a matchgate (G, X), let $\mathrm{PerfMatch}(G - Z)$ be the sum, over all perfect matchings M of $G - Z$, of the product of the weights of matching edges in M (If all weights are 1, this is the number of perfect matchings.), where Z is the subset of the output nodes having the characteristic sequence $\chi_Z = i_1 i_2 \cdots i_n$, $G - Z$ is the graph obtained from G by removing Z and its adjacent edges.

Each matchgate is assigned a standard signature. A generator Γ with n output nodes is assigned a standard signature $\underline{G} = (\underline{G}^{i_1 i_2 \cdots i_n})$, $i_1, i_2, \cdots, i_n \in \{0, 1\}$, where

$$\underline{G}^{i_1 i_2 \cdots i_n} = \mathrm{PerfMatch}(G - Z).$$

$\underline{G}$ can be viewed as a column vector of dimension 2^n.

Similarly, a recognizer Γ' with n input nodes is assigned a standard signature $\underline{R} = (\underline{R}_{i_1 i_2 \cdots i_n})$, $i_1, i_2, \cdots, i_n \in \{0, 1\}$, where

$$\underline{R}_{i_1 i_2 \cdots i_n} = \mathrm{PerfMatch}\big(G' - Z\big).$$

$\underline{R}$ can be viewed as a row vector of dimension 2^n.

Generators and recognizers are essentially the same as far as their standard signatures are concerned. The distinction is how they transform with respect to a basis transformation over some field (the default is $\mathbf{C}$).

A *basis* M on domain size k is a $2^\ell \times k$ matrix $(\alpha_1, \alpha_2, \cdots, \alpha_k)$, where α_i has dimension 2^ℓ (size ℓ). Under a basis M, we can talk about the signature of a matchgate after the transformation.

Definition 2.1. A generator Γ has signature G (written as a column vector) under basis M iff $M^{\otimes n} G = \underline{G}$ is the standard signature of the generator Γ.

Definition 2.2. A recognizer Γ' has signature R (written as a row vector) under basis M iff $\underline{R} M^{\otimes n} = R$ where $\underline{R}$ is the standard signature of the recognizer Γ'.

Definition 2.3. A column vector G (resp. a row vector R) is realizable over a basis M iff there exists a generator Γ (resp. a recognizer Γ') such that G (resp. R) is the signature of Γ (resp. Γ') under basis M. They are simultaneously realizable if they are realizable over a common basis.

A matchgrid $\Omega = (A, B, C)$ is a weighted planar graph consisting of a disjoint union of: a set of g generators $A = (A_1, A_2, \cdots, A_g)$, a set of r recognizers $B = (B_1, B_2, \cdots, B_r)$, and a set of f connecting edges $C = (C_1, C_2, \cdots, C_f)$, where each C_i edge has weight 1 and joins an output node of a generator with an input node of a recognizer, so that every input and output node in every constituent matchgate has exactly one such incident connecting edge.

Let $G(A_i, M)$ be the signature of generator A_i under the basis M and $R(B_j, M)$ be the signature of recognizer B_j under the basis M. Let $G = \bigotimes_{i=1}^{g} G(A_i, M)$ and $R = \bigotimes_{j=1}^{r} R(B_j, M)$ be their tensor product, then $\mathrm{Holant}(\Omega)$ is defined to be the *contraction* of these two product tensors (the sum over all indices of the product of the corresponding values of G and R), where the corresponding indices match up according to the f connecting edges in C.

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[3] Following [10], to allow greater flexibility in the design of holographic algorithms, a basis here may not be linearly independent, e.g., when $\ell = 1$, $k = 3$. However to be applicable to matchgates, the number of rows must be a power of 2.