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Minimum neighborhood in a generalized cube

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Abstract

Generalized cubes are a subclass of hypercube-like networks, which include some hypercube variants as special cases. Let $\theta_G(k)$ denote the minimum number of nodes adjacent to a set of k vertices of a graph G. In this paper, we prove $\theta_G(k) \geqslant -\frac{1}{2}k^2 + (2n - \frac{3}{2})k - (n^2 - 2)$ for each n-dimensional generalized cube and each integer k satisfying $n + 2 \leqslant k \leqslant 2n$. Our result is an extension of a result presented by Fan and Lin [J. Fan, X. Lin, The t/k-diagnosability of the BC graphs, IEEE Trans. Comput. 54 (2) (2005) 176–184].

Keywords: Interconnection networks; Hypercube; Generalized cube; Minimum neighborhood problem

1. Introduction

Due to its attractive properties such as symmetry, recursive structure, small node degree, small diameter, and simple yet optimal routing and broadcasting algorithms, the hypercube structure has long been one of the most popular schemes for interconnecting processors in multicomputer systems [12]. The communication delay of an interconnection network is a fundamental concern, which is commonly measured by its diameter; usually, a network with small diameter is preferred. Hypercube, however, does not enjoy the smallest diameter for its resources. To achieve smaller diameter with the same numbers of nodes and links, a number of hypercube variants were proposed [2,4,6,7,14,20].

As a generalization of hypercube networks, the *hypercube-like networks* have received considerable attention [5,8,15,17]. Recently, Park and Chwa [13] and Fan and Lin [9] studied a subclass of hypercube-like networks named as the *BC graphs*, which include some hypercube variants [2,4,6,7,14,20] as special cases. In this paper, we referred to BC graphs as *generalized cubes* to stress their similarity to hypercube.

For a graph G, let $\theta_G(k)$ denote the minimum number of nodes adjacent to a set of k vertices of G. In this paper, we consider the following problem:

Minimum neighborhood problem (MNP). Given a graph G and an integer k with $1 \leqslant k \leqslant |V(G)|$, to determine $\theta_G(k)$.

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The MNP not only is of interest in its own right, but the obtained result is useful in system-level fault diagnosis [3,9,11,16] and in the analysis of fault tolerance of interconnection networks [18,19]. Somani and Peleg [16] presented the following result for an n-dimensional cube G:

$$\theta_G(k) = -\frac{1}{2}k^2 + \left(n - \frac{1}{2}\right)k + 1 \quad \text{for } 1 \leqslant k \leqslant n+1.$$

Recently, Fan and Lin [9] reported the following result for an n-dimensional generalized cube G:

$$\theta_G(k) \geqslant -\frac{1}{2}k^2 + \left(n - \frac{1}{2}\right)k + 1 \quad \text{for } 1 \leqslant k \leqslant n+1.$$

For other related work the reader is referred to Refs. [1,21].

This paper studies the MNP for an n-dimensional generalized cube G. The main contribution of this paper is as follows:

$$\theta_G(k) \geqslant -\frac{1}{2}k^2 + \left(2n - \frac{3}{2}\right)k - (n^2 - 2) \quad \text{for } n+2 \leqslant k \leqslant 2n.$$

Moreover, this inequality is optimal in the sense that for an n-dimensional cube G,

$$\theta_G(k) = -\frac{1}{2}k^2 + \left(2n - \frac{3}{2}\right)k - (n^2 - 2) \quad \text{for } n+2 \leqslant k \leqslant 2n.$$

The result presented here extends the one in [9]. As a result, our result applies to various hypercube variants, including the twisted cube [2], the crossed cube [7], the Möbius cube [6], and the locally twisted cube [20].

The remaining materials are organized in this fashion: Preliminary knowledge is provided in Section 2. The main results of this paper are presented and proven in Section 3. Some remarks are made in Section 4.

2. Preliminaries

In this paper, we use a graph $G = (V(G), E(G))$ to represent an interconnection network, where nodes and edges represent processors and communication links between processors, respectively. For a set S of nodes of graph G, let $N_G(S) = \{v \in V(G) - S\colon$ there is $u \in S$ such that $(u, v) \in E(G)\}$, $N_G[S] = N_G(S) \cup S$. For our purpose, let us introduce the following notation:

$$\theta_G(k) = \min\bigl\{|N_G(S)|\colon S \text{ is a set of } k \text{ nodes of graph } G\bigr\}.$$

An n-dimensional hypercube is a graph with 2^n nodes each node of which is identified with a unique binary sequence of length n so that two nodes are adjacent if and only if they differ in exactly one bit position. For other fundamental graph-theoretic terminology the reader is referred to Ref. [10].

Definition 2.1 [9,13]. An n-dimensional *generalized cube* is defined recursively in the following fashion:

(1) A one-dimensional generalized cube is a graph on a single node.
(2) For $n \geqslant 2$, an n-dimensional generalized cube G is built from two node-disjoint $(n-1)$-dimensional generalized cubes G_1 and G_2 by adding a set of edges $M_\phi = \{(u, \phi(u))\colon u \in V(G_1)\}$, where ϕ is a bijective mapping from $V(G_1)$ to $V(G_2)$. A generalized cube G so constructed is denoted by $G = (G_1, G_2; \phi)$. For a node $u \in V(G_1)$, $\phi(u)$ is called as the ϕ-neighbor of u, and u is called as the ϕ^{-1}-neighbor of $\phi(u)$.

In the sense of isomorphism, the length-4 cycle is the unique two-dimensional generalized cube. In the sense of isomorphism, there are only two three-dimensional generalized cubes, which are shown in Fig. 1. Fig. 2 shows five typical four-dimensional generalized cubes.

For the purpose of estimating $\theta_G(k)$ for an n-dimensional generalized cube G, two families of quadratic functions, $\{p_n(x)\}_{n\geqslant 1}$ and $\{q_n(x)\}_{n\geqslant 1}$, are necessary, which are defined as

$$p_n(x) = -\frac{1}{2}x^2 + \left(n - \frac{1}{2}\right)x + 1,$$
$$q_n(x) = -\frac{1}{2}x^2 + \left(2n - \frac{3}{2}\right)x - (n^2 - 2).$$

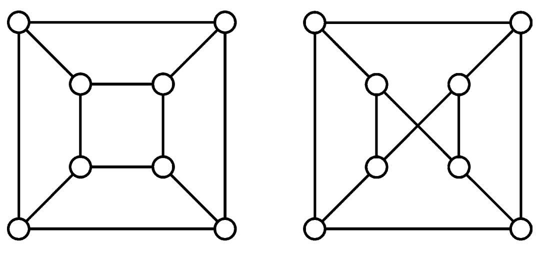

Fig. 1. Two three-dimensional generalized cubes.