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Density of straight-line 1-planar graph drawings

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ABSTRACT

A 1-*planar drawing* of a graph is such that each edge is crossed at most once. In 1997, Pach and Tóth showed that any 1-planar drawing with n vertices has at most $4n-8$ edges and that this bound is tight for $n \geqslant 12$. We show that, in fact, 1-planar drawings with n vertices have at most $4n-9$ edges, if we require that the edges are straight-line segments. We also prove that this bound is tight for infinitely many values of $n \geqslant 8$. Furthermore, we investigate the density of 1-planar straight-line drawings with additional constraints on the vertex positions. We show that 1-planar drawings of bipartite graphs whose vertices lie on two distinct horizontal layers have at most $1.5n-2$ edges, and we prove that 1-planar drawings such that all vertices lie on a circumference have at most $2.5n-4$ edges; both these bounds are also tight.

1. Introduction

Drawings of graphs with few crossings per edge have been studied in graph drawing, graph theory, and computational geometry. A k-*planar drawing* of a graph is such that each edge is crossed at most k times. Planar drawings can be regarded as k-planar drawings for $k=0$.

Pach and Tóth [11] showed that k-planar drawings with n vertices cannot have more than $4.108\sqrt{kn}$ edges. For $k \leqslant 4$, they also established a better bound, $(k+3)(n-2)$, and proved that this bound is tight for $k \leqslant 2$. Hence, 1-planar drawings cannot have more than $4n-8$ edges, and Pach and Tóth proved that for any $n \geqslant 12$ there exists a 1-planar drawing with n vertices and $4n-8$ edges (i.e., the $4n-8$ bound is tight for $n \geqslant 12$).

Maximal 1-planar graphs have been also investigated. A 1-planar graph G is maximal if it admits a 1-planar drawing and no edge can be added to G without loosing this property. Brandenburg et al. [2] showed that there exist maximal 1-planar graphs with only $2.65n + O(1)$ edges and that every maximal 1-planar graph has at least $2.1n - O(1)$ edges. Korzhik and Mohar [10] showed that testing 1-planarity is NP-hard and Auer et al. [1] proved that this is true even if a *rotation system* for the graph is given and cannot be changed (a rotation system fixes for each vertex the clockwise ordering of its incident edges). On the positive side, Eades et al. [7] proved that maximal 1-planarity can be tested in linear time for a graph with a given rotation system.

The relationships between 1-planar drawings and *right angle crossing* drawings have been studied in [8]. In a right angle crossing drawing, edges cross only at right angles (see, e.g., [5,6] for definitions and results about right angle crossing drawings).

In this paper we concentrate on 1-planar drawings with straight-line edges. We prove that, in fact, a straight-line 1-planar drawing with n vertices has at most $4n-9$ edges, and that this bound is tight. More precisely, we show infinitely many graphs with $n \geqslant 8$ vertices and $4n-9$ edges that are straight-line 1-planar drawable. This result and the bound proved in [11] immediately imply that there are infinitely many graphs that admit a 1-planar drawing but that do not admit a straight-line 1-planar drawing (i.e., all 1-planar graphs with $4n-8$ edges). In this respect, it is worth recalling that Hong et al. [9] characterized those 1-planar drawings whose edges can be

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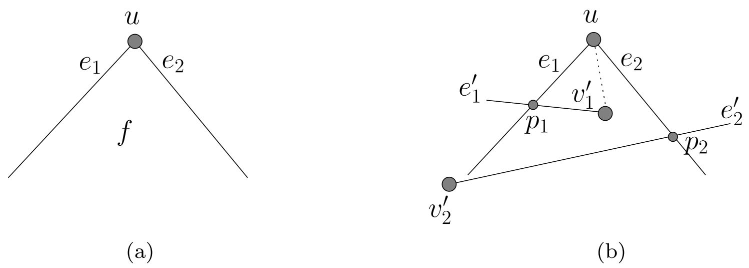

Fig. 1. Illustration for the proof of Lemma 1.

"straightened" keeping unchanged their edge crossings and their rotation system; they are those 1-planar drawings that do not contain specific forbidden structures; our result implies that every 1-planar drawing with n vertices and $4n-8$ edges necessarily contains at least one of the forbidden structures described in [9].

Furthermore, we investigate the edge density of 1-planar straight-line drawings with additional constraints on the vertex positions. We show that 1-planar drawings of bipartite graphs whose vertices lie on two distinct horizontal layers have at most $1.5n-2$ edges (which follows easily from a result in [3]), and we prove that 1-planar drawings such that all vertices lie on a circumference have at most $2.5n-4$ edges; both these bounds are tight.

The remainder of the paper is structured as follows. The bound on the number of edges of general straight-line 1-planar drawings is proved in Section 2. The edge density of constrained straight-line 1-planar drawings is discussed in Section 3. Conclusions and open problems are in Section 4.

2. General straight-line 1-planar drawings

To prove that a straight-line 1-planar drawing has at most $4n-9$ edges, we use a technique similar to that applied to derive the $4n-10$ bound on the density of a straight-line right angle crossing drawing (also called a *RAC drawing*) [4,5]. We recall that a RAC drawing is a drawing where the edges can only cross at right angles, i.e., any two crossing edges are orthogonal to each other. Eades and Liotta recently investigated some relationships between 1-planar straight-line drawings and straight-line RAC drawings [8]. They show that every graph admitting a straight-line RAC drawing that is maximally dense (i.e., with $4n-10$ edges) also admits a straight-line 1-planar drawing; they also prove that there exist infinitely many graphs with $4n-10$ edges that admit a straight-line 1-planar drawing but that have no straight-line RAC drawing.

Let G be a graph that admits a 1-planar straight-line drawing D. Each edge of G is either crossing free in D, or it forms a crossing with exactly one other edge in the drawing. Hence, we can color the edges of G with three colors, red, blue, and green, such that a red edge does not cross any other edge in D and each blue edge crosses (exactly) one green edge in D. We denote by G_r the subgraph of G consisting of all vertices of G and only its red edges. Similarly, G_b (resp. G_g) is the subgraph of G consisting of all vertices of G and only its blue edges (resp. the green edges). Graphs G_r, G_b, and G_g are called the *red subgraph*, the *blue subgraph*, and the *green subgraph* of G induced by D, respectively. The subdrawings of D for the red, the blue, and the green subgraphs are denoted by D_r, D_b, and D_g, respectively. By definition, D_r, D_b, and D_g are planar drawings. Also, if $D_{rb} = D_r \cup D_b$ and $D_{rg} = D_r \cup D_g$, we also have that D_{rb} and D_{rg} are planar.

A graph G is a *maximal straight-line* 1*-planar graph* if: (i) G admits a straight-line 1-planar drawing D; (ii) the graph obtained from G by adding any other edge has no straight-line 1-planar drawing.

Similarly to straight-line RAC drawings [4,5], we are able to prove the following technical lemma.

Lemma 1. *Let G be a maximal straight-line 1-planar graph and let D be a straight-line 1-planar drawing of G. Let D_r, D_b, and D_g be the subdrawings of D for the red, the blue, and the green subgraph of G, respectively. Drawings D_{rb} and D_{rg} have the same external face, which consists of red edges only. Also, every internal face of D_{rb} (resp. of D_{rg}) contains at least two red edges.*

Proof. Consider first the (not necessarily simple) polygon $P(D)$ formed by the sequence of vertices, crossing points, and edge segments encountered while walking on the external contour of D (i.e., the contour delimiting D). $P(D)$ must be a convex polygon, otherwise it would be possible to add at least one extra red edge to the convex-hull of $P(D)$ between two vertices of $P(D)$ that are also vertices of G; this would contradict the fact that G is maximal straight-line 1-planar. Since the segments of a convex polygon cannot cross, it follows that $P(D)$ is formed only by vertices and edges of G, and all its edges are red edges (because they do not cross). This immediately implies that $P(D)$ coincides with the external face of D_{rb} and of D_{rg}.

We now concentrate on the internal faces of D_{rb} and of D_{rg}. Let f be an internal face of the red–blue drawing (the proof is the same for the internal faces of the red–green drawing). The boundary of f is a polygon (not necessarily simple) and it must have at least three vertices with an interior angle smaller than 180°. Let u be any of these vertices, and let e_1 and e_2 be the two edges incident to u on the boundary of f that form an angle smaller than 180° inside f (see Fig. 1(a)). Below, we prove that at least one of e_1 and e_2 is red. Since there are at least three vertices with an interior angle smaller than 180°, and since any two of these vertices share at most one edge of the boundary of f, this suffices to prove that f has at least two red edges.

Suppose by contradiction that both e_1 and e_2 are blue edges. This implies that e_1 is crossed by a green edge e'_1