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On behavioural pseudometrics and closure ordinals

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ABSTRACT

A behavioural pseudometric is often defined as the least fixed point of a monotone function F on a complete lattice of 1-bounded pseudometrics. According to Tarski's fixed point theorem, this least fixed point can be obtained by (possibly transfinite) iteration of F, starting from the least element $\perp$ of the lattice. The smallest ordinal α such that $F^\alpha(\perp) = F^{\alpha+1}(\perp)$ is known as the closure ordinal of F. We prove that if F is also continuous with respect to the sup-norm, then its closure ordinal is ω. We also show that our result gives rise to simpler and modular proofs that the closure ordinal is ω.

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1. Introduction

A behavioural equivalence addresses the fundamental question whether two states of a system behave the same. Numerous behavioural equivalences have been proposed (see, for example, [1] for an overview). For systems that contain quantitative data such as time or probabilities, a quantitative generalization of a behavioural equivalence is more appropriate. In such a setting one is interested how similar two states behave. A *behavioural pseudometric*, first introduced in [2], is such a quantitative generalization. It assigns to every pair of states a distance, that is, a real number in the interval $[0, 1]$. The smaller the distance, the more alike the states behave. If their distance is zero, then the states behave exactly the same, that is, they are behaviourally equivalent. Numerous behavioural pseudometrics have been proposed (see, for example, [3–6]).

Consider a system with a set of states X. A behavioural pseudometric for such a system is a pseudometric on X. Often, the behavioural pseudometric is defined as the *least fixed point* of a monotone function F on a complete lattice of all 1-bounded pseudometric spaces on X (see, for example, [7]). According to Tarski's fixed point theorem [8], this least fixed point can be obtained iteratively. Starting from the least element $\perp$ of the lattice, the function F is applied repeatedly. The smallest ordinal α such that $F^\alpha(\perp) = F^{\alpha+1}(\perp)$ is called the *closure ordinal* of F.

In our setting, the least element of the lattice is the pseudometric d_0 that assigns to each pair of states distance zero, that is, all states are considered behaviourally equivalent. The function F applied to $F^\alpha(d_0)$ increases some of the distances of $F^\alpha(d_0)$. This can be viewed as a quantitative generalization of partition refinement. As in partition refinement, where a fixed point is reached if no further refinements are needed, a fixed point of F is reached when no distances are further increased.

If the closure ordinal of F is ω, then an iterative algorithm to approximate the behavioural pseudometric may be feasible. An example of such an iterative algorithm can be found in, for example, [9]. Furthermore, properties of the behavioural pseudometric may be proved using a simple inductive argument if the closure ordinal of F is ω. An example of such a proof can be found in, for example, [10, Appendix B].

In the literature, several (somewhat ad hoc) proofs can be found that the monotone function F defining a behavioural pseudometric has closure ordinal ω (see, for

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example, [10,11]). In Section 2 we show that if F is also continuous with respect to the sup-norm and the set X is finite, then the closure ordinal of F is ω. It is often easier to prove that F is nonexpansive, which implies continuity, with respect to the sup-norm. As shown by means of an example in Section 3, this result amounts to a fairly simple proof that the closure ordinal of F is ω. Furthermore, as shown in Section 4 by means of an example, this result allows for modular proofs. In particular, if F is defined as the composition of G and H, it suffices to prove that G and H are (monotone and) nonexpansive.

2. The main result

We fix a *finite* set X. Recall that a 1-*bounded pseudometric* on X is a function $d : X \times X \to [0,1]$ satisfying for all $x_1, x_2, x_3 \in X$,

- $d(x_1, x_1) = 0$,
- $d(x_1, x_2) = d(x_2, x_1)$ and
- $d(x_1, x_3) \leqslant d(x_1, x_2) + d(x_2, x_3)$.

We denote the set of 1-bounded pseudometrics on X by $D(X)$. This set can be turned into a complete lattice by means of the pointwise order: $d_1 \sqsubseteq d_2$ if $d_1(x_1, x_2) \leqslant d_2(x_1, x_2)$ for all $x_1, x_2 \in X$ (see, for example, [7, Lemma 3.2] for a proof). The least element $\perp$ of this lattice assigns distance zero to each pair of states. Let $F : D(X) \to D(X)$. For $n \in \omega$, we define the pseudometric d_n on X by

$$d_n = \begin{cases} \perp & \text{if } n = 0, \\ F(d_{n-1}) & \text{otherwise.} \end{cases}$$

The pseudometric d_ω on X is defined by $d_\omega = \bigsqcup_{n\in\omega} d_n$. Hence, $d_\omega(x_1, x_2) = \sup_{n\in\omega} d_n(x_1, x_2)$. Recall that F is *monotone* if $d_1 \sqsubseteq d_2$ implies $F(d_1) \sqsubseteq F(d_2)$ for all $d_1, d_2 \in D(X)$.

Proposition 1. *If $F : D(X) \to D(X)$ is monotone then for all $n \in \omega$,*

1. $d_n \sqsubseteq d_{n+1}$ *and*
2. $d_n \sqsubseteq F(d_\omega)$.

Proof.

1. We prove this part by induction on n. In the base case, obviously $d_0 = \perp \sqsubseteq d_1$. In the inductive case, assume that $d_n \sqsubseteq d_{n+1}$. Since F is monotone, $d_{n+1} = F(d_n) \sqsubseteq F(d_{n+1}) = d_{n+2}$.
2. By definition, $d_n \sqsubseteq \bigsqcup_{n\in\omega} d_n = d_\omega$ for all $n \in \omega$. Since F is monotone, $d_{n+1} = F(d_n) \sqsubseteq F(d_\omega)$. Obviously, also $d_0 = \perp \sqsubseteq F(d_\omega)$. □

The set of real valued functions on $X \times X$, which is a superset of $D(X)$, can be turned into a Banach space by means of the sup-norm: $\|f\| = \max_{x_1,x_2\in X} |f(x_1, x_2)|$. Recall that a function $F : D(X) \to D(X)$ is *continuous* if for all $\epsilon > 0$ there exists $\delta > 0$ such that $\|d_1 - d_2\| < \delta$ implies $\|F(d_1) - F(d_2)\| < \epsilon$ for all $d_1, d_2 \in D(X)$.

Theorem 1. *If $F : D(X) \to D(X)$ is monotone and continuous, then the closure ordinal of F is ω.*

Proof. First, we show that $d_\omega \sqsubseteq F(d_\omega)$. By Proposition 1.2, $F(d_\omega)$ is an upper bound of $\{d_n \mid n \in \omega\}$. Since d_ω is its least upper bound by definition, $d_\omega \sqsubseteq F(d_\omega)$.

Next, we show that $(d_n)_{n\in\omega}$ converges to d_ω. It suffices to show that

$$\forall \epsilon > 0\colon \exists m \in \omega\colon \forall n \geqslant m\colon \|d_\omega - d_n\| \leqslant \epsilon.$$

Let $\epsilon > 0$ and $x_1, x_2 \in X$. By definition, $d_\omega(x_1, x_2) = \sup_{n\in\omega} d_n(x_1, x_2)$. Hence, from Proposition 1.1 we can conclude that

$$\exists m_{x_1,x_2} \in \omega\colon \forall n \geqslant m_{x_1,x_2}\colon \left|d_\omega(x_1, x_2) - d_n(x_1, x_2)\right| \leqslant \epsilon.$$

Hence,

$$\forall n \geqslant \max\{m_{x_1,x_2} \mid x_1, x_2 \in X\}\colon \|d_\omega - d_n\| \leqslant \epsilon.$$

That is, $(d_n)_{n\in\omega}$ converges to d_ω.

Since F is continuous, we can deduce from the above that $(F(d_n))_{n\in\omega}$ converges to $F(d_\omega)$ and, hence, $(d_n)_{n\in\omega}$ converges to $F(d_\omega)$. That is,

$$\forall \epsilon > 0\colon \exists m \in \omega\colon \forall n \geqslant m\colon \left\|F(d_\omega) - d_n\right\| \leqslant \epsilon.$$

Hence, for all $x_1, x_2 \in X$, $|F(d_\omega)(x_1, x_2) - d_n(x_1, x_2)| \leqslant \epsilon$. Therefore, from Proposition 1.2 we can conclude that $F(d_\omega)(x_1, x_2) \leqslant d_n(x_1, x_2) + \epsilon$. Consequently, $F(d_\omega)(x_1, x_2) \leqslant \sup_{n\in\omega} d_n(x_1, x_2)$. Hence, $F(d_\omega) \sqsubseteq d_\omega$. □

Rather than proving that F is continuous, we can often prove an even stronger property, namely that F is *nonexpansive*, that is, for all $d_1, d_2 \in D(X)$, $\|F(d_1) - F(d_2)\| \leqslant \|d_1 - d_2\|$. Since F is also assumed to be monotone, it suffices to prove the following.

Corollary 1. *If $F : D(X) \to D(X)$ is monotone and for all $d_1, d_2 \in D(X)$, $d_1 \sqsupseteq d_2$ implies that for all $x_1, x_2 \in X$,*

$$F(d_1)(x_1, x_2) - F(d_2)(x_1, x_2) \leqslant \|d_1 - d_2\|, \tag{1}$$

then the closure ordinal of F is ω.

Proof. It suffices to show that (1) implies that F is nonexpansive and, hence, continuous. Assume (1) and let $d_1, d_2 \in D(X)$. Without loss of generality, suppose that $d_1 \sqsupseteq d_2$. Since F is monotone, $F(d_1) \sqsupseteq F(d_2)$. Hence,

$$\begin{aligned} &\left\|F(d_1) - F(d_2)\right\| \\ &\quad = \sup_{x_1,x_2\in X} \left|F(d_1)(x_1, x_2) - F(d_2)(x_1, x_2)\right| \\ &\quad = \sup_{x_1,x_2\in X} F(d_1)(x_1, x_2) - F(d_2)(x_1, x_2) \\ &\qquad [F(d_1) \sqsupseteq F(d_2)] \\ &\quad \leqslant \|d_1 - d_2\| \quad [(1)]. \end{aligned}$$

Therefore, F is nonexpansive. □

Theorem 1 can also be obtained as a corollary of a more general result about Banach lattices. Recall that the set of real-valued functions on $X \times X$ forms a real vector space.

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