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# $(\Delta+1)$-total-colorability of plane graphs of maximum degree $\Delta \geqslant 6$ with neither chordal 5-cycle nor chordal 6-cycle ☆

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ABSTRACT

In this paper, we prove that a plane graph of maximum degree $\Delta \geqslant 6$ is $(\Delta+1)$-totally-colorable if it contains neither chordal 5-cycle nor chordal 6-cycle. This further extends the known class of plane graphs of maximum degree $\Delta$ in which every graph is $(\Delta+1)$-totally-colorable.



## 1. Introduction

All graphs considered here are finite, simple and undirected. Undefined terminology and notation in this paper can be found in the book by Bondy and Murty [2]. A graph is *planar* if it can be embedded into the plane so that its edges meet only at their ends. Any such embedding of a planar graph is called a *plane* graph. For a plane graph $G$, we denote its vertex set, edge set, face set, maximum degree and minimum degree by $V(G)$, $E(G)$, $F(G)$, $\Delta(G)$ and $\delta(G)$ (or simply $V$, $E$, $F$, $\Delta$ and $\delta$ if this causes no confusion), respectively. A *k-cycle* is a cycle of length $k$. A *chord* of a cycle is an edge connecting two non-consecutive vertices of the cycle. A *chordal k-cycle* is a $k$-cycle with at least one chord. A *forest* is a graph without cycles.

A *k-total-coloring* of a graph $G$ is a mapping $\phi$ from $V \cup E$ to the set of available colors $\{1, 2, \dots, k\}$ such that $\phi(x) \neq \phi(y)$ for every pair of adjacent or incident elements $x, y \in V \cup E$. $G$ is called *k-totally-colorable* if it admits a $k$-total-coloring.

Clearly, at least $(\Delta+1)$ colors are needed to color $G$ totally. Vizing [22] and Behzad [1] independently conjectured that every graph is $(\Delta+2)$-totally-colorable. This conjecture is called Total Coloring Conjecture, in short, TCC. TCC has been extensively studied. However, even for plane graphs, TCC remains open. It is unknown whether every plane graph of maximum degree 6 is 8-totally-colorable. More precisely, with one exception $\Delta = 6$, it has been proved that every plane graph of maximum degree $\Delta$ is $(\Delta+2)$-totally-colorable, see [15,21,11,4,16,12,28]. As for $\Delta = 6$, it has been proved that a plane graph of maximum degree 6 is 8-totally-colorable if it does not contain any 4-cycle [25]; or 5-cycle [9,10]; or 6-cycle [9,10].

It is interesting to notice that every plane graph of large enough maximum degree $\Delta$ is $(\Delta+1)$-totally-colorable. This fact was first established in [3] for $\Delta \geqslant 16$, which was later improved successively to $\Delta \geqslant 14$ in [4], $\Delta \geqslant 12$ in [6], $\Delta \geqslant 11$ in [5], $\Delta \geqslant 10$ in [23], and $\Delta \geqslant 9$ in [13]. It is challenging to determine the $(\Delta+1)$-total-colorability neatly as in [3,4,6,5,23,13] for plane graphs of smaller maximum degree $\Delta$ (that is, $\Delta \leqslant 8$). It seems wise to study the

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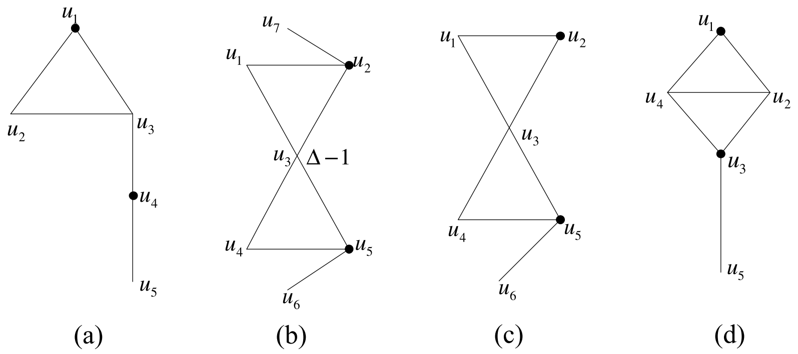


**Fig. 1.** Forbidden configurations in $G$.

$(\Delta+1)$-total-colorability for plane graphs of smaller maximum degree $\Delta$ under some additional restrictions. It has been proved that a plane graph of maximum degree 8 is 9-totally-colorable if it does not contain any chordal 5-cycle [17]; or chordal 6-cycle [17]; or intersecting triangles [18]; or adjacent triangles [8], and that a plane graph of maximum degree 7 is 8-totally-colorable if it does not contain any 3-cycle [6]; or 4-cycle [24]; or 5-cycle [20]; or 6-cycle [26]; or intersecting 4-cycles [14]; or intersecting 3-cycles [27]. As for the 7-total-colorability of plane graphs of maximum degree 6, it has first been proved in [7] that a plane graph of maximum degree 6 is 7-totally-colorable if it contains neither 3-cycle nor 4-cycle. This is recently improved in [19] to that a plane graph of maximum degree 6 is 7-totally-colorable if it does not contain any 4-cycle. In [19], the authors conjectured that every plane graph of maximum degree $\Delta \geqslant 4$ is $(\Delta+1)$-totally-colorable. This conjecture may be called PTCC. This paper offers PTCC a new evidence as follows:

**Theorem 1.** *Every plane graph of maximum degree $\Delta \geqslant 6$ is $(\Delta+1)$-totally-colorable if it contains neither chordal 5-cycle nor chordal 6-cycle.*

Let $M = \max\{7, \Delta+1\}$. Instead of Theorem 1, we shall prove a stronger result:

**Theorem 2.** *Every plane graph of maximum degree $\Delta$ is $M$-totally-colorable if it contains neither chordal 5-cycle nor chordal 6-cycle.*

## 2. Lemmas

Let $G = (V, E, F)$ be a counterexample to Theorem 2 with $\sigma(G) = |V| + |E|$ as small as possible. By the choice of $G$, any proper subgraph of $G$ is $M$-totally-colorable while $G$ itself is not. This is implicitly used later to establish the structural properties of $G$. It is easy to see that $G$ is 2-connected. Hence, $\Delta \geqslant 2$ and the boundary of every face in $G$ is a cycle. Call $v \in V$ a *k-vertex* or a *$k^+$-vertex* or a *$k^-$-vertex* if $d(v)$, the degree of $v$ in $G$, is equal to or at least or at most $k$, respectively. A $k$-vertex adjacent to $v$ is called a *k-neighbor* of $v$. A *$(k,l)$-edge* is an edge connecting a $k$-vertex to an $l$-vertex. For a face $f \in F$, we define its *degree* $d(f)$ to be the length of the boundary cycle of $f$. The notion of a $k$-face, a $k^+$-face, or a $k^-$-face is defined in a way analogous to that for vertices. A face $f$ with the boundary cycle $C = v_1v_2 \ldots v_kv_1$ is often written as $f = [v_1 \ldots v_k]$, and called a *$(d(v_1), d(v_2), \ldots, d(v_k))$-face.*

Note that TCC has been proved for $\Delta \leqslant 5$, see [15,21,11,12,28]. Hence the graph $G$, as a counterexample of Theorem 2, has the maximum degree $\Delta \geqslant 6$. Thus the value of $M$ in Theorem 2 is equal to $\Delta+1$ for some $\Delta \geqslant 6$. Lemmas 1–6 below have already been known for large enough values of $\Delta$, and their proofs work also when $\Delta = 6$. We prove them here for completeness.

**Lemma 1.** *Let $uv \in E$. If $d(u) \leqslant \lfloor \frac{\Delta}{2} \rfloor$, then $d(u)+d(v) \geqslant \Delta+2$. As corollaries, the two neighbors of a 2-vertex are $\Delta$-vertices; the three neighbors of a 3-vertex are $(\Delta-1)^+$-vertices.*

**Proof.** If $G$ contains an edge $uv$ with $d(u) \leqslant \lfloor \frac{\Delta}{2} \rfloor$ and $d(u)+d(v) \leqslant \Delta+1$, then, by the choice of $G$, we can totally color $G-uv$ with $(\Delta+1)$ colors. Erase the color on $u$, then $uv$ and $u$ can be properly colored in turn, since the number of forbidden colors is at most $(d(u)-1)+(d(v)-1)+1 \leqslant (\Delta+1)-1 = \Delta$ for $uv$ and $\lfloor \frac{\Delta}{2} \rfloor + \lfloor \frac{\Delta}{2} \rfloor \leqslant \Delta$ for $u$. Thus we obtain a $(\Delta+1)$-total-coloring of $G$, a contradiction. □

**Lemma 2.** *The subgraph induced by all $(2,\Delta)$-edges in $G$ is a forest. In particular, $G$ has no $(2,\Delta,2,\Delta)$-face.*

**Proof.** Otherwise, by Lemma 1, the graph $G$ contains a cycle of even length, say $C$, in which every edge connects a 2-vertex to a $\Delta$-vertex. Deleting all 2-vertices of $C$ in $G$, we get a new graph $G'$ from $G$. By the choice of $G$, $G'$ admits a $(\Delta+1)$-total-coloring. This coloring can be extended to $G$ by first properly coloring all edges of $C$ (since an even cycle is list-2-edge-colorable) and then properly coloring all 2-vertices of $C$. This is a contradiction completing the proof. □

In Fig. 1, the vertices marked by • have no other neighbors in $G$. In order to prove that $G$ contains none of the four configurations depicted in Fig. 1, we always first suppose that $G$ contains the configuration under consideration, then choose an edge $e$ and remove it from $G$, getting a new graph $G' = G - e$. By the choice of the graph $G$, $G'$ admits a $(\Delta+1)$-total-coloring $\phi : V(G') \cup E(G') \to \{1, 2, \ldots, \Delta+1\}$. If we can get a $(\Delta+1)$-total-coloring of