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Approximating integer programs with positive right-hand sides

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ABSTRACT

We study minimisation of integer linear programs with positive right-hand sides. We show that such programs can be approximated within the maximum absolute row sum of the constraint matrix A whenever the variables are allowed to take values in $\mathbb{N}$. This result is optimal under the unique games conjecture. When the variables are restricted to bounded domains, we show that finding a feasible solution is **NP**-hard in almost all cases.

1. Introduction

We study the approximability of minimising integer linear programs with positive right-hand sides. Let n and m be positive integers, representing the number of variables and the number of inequalities, respectively. Let $\mathbf{x}^T = (x_1, \dots, x_n)$ be a vector of n variables, A be an integer $m \times n$ matrix, $\mathbf{b} \in (\mathbb{Z}^+)^m$, and $\mathbf{c} \in (\mathbb{Q}^+ \cup \{0\})^n$. Finally, let X be some given subset of $\mathbb{N}^n$. We consider here various restrictions of the following integer linear program:

$$\begin{aligned} \text{Minimise} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} \geqslant \mathbf{b}, \\ & \mathbf{x} \in X. \end{aligned} \qquad \text{(IP)}$$

Typically, X is either $\mathbb{N}^n$ or $\{\mathbf{x} \in \mathbb{Z}^n \mid \mathbf{0} \leqslant \mathbf{x} \leqslant \mathbf{d}\}$ for some $\mathbf{d} \in (\mathbb{Z}^+)^n$, where the inequalities are to hold componentwise. A commonly occurring instance of the latter case is when $X = \{0,1\}^n$, so-called 0–1 programming. In all but very restricted cases, (IP) is **NP**-hard to solve to optimality. Instead, the effort is directed towards finding approximation algorithms and improving the bound within which it is possible to find approximate solutions. Formally, a minimisation problem Π is said to be *approximable within* (a real constant) $c \geqslant 1$ if there exists a polynomial time algorithm A such that for all instances x of Π, $A(x)/\textsc{Opt}(x) \leqslant c$.

Let $\mathbf{a}_j^T = (a_{j1}, \dots, a_{jn}) \in \mathbb{Z}^n$ be the jth row of A. We will use the norm $\|\mathbf{a}_j\|_1 = \sum_{i=1}^n |a_{ji}|$ as well as the *maximum absolute row sum norm* of A, defined as $\|A\|_\infty = \max_{1 \leqslant j \leqslant m} \|\mathbf{a}_j\|_1$. Let $(\text{IP})_k$ denote the subset of (IP) where $\|A\|_\infty \leqslant k$. We show that $(\text{IP})_k$ can unconditionally be approximated within k when $X = \mathbb{N}^n$, but cannot be approximated within $k - \epsilon$, $\epsilon > 0$, if Khot's *unique games conjecture* holds [9]. We also show that finding a feasible solution to (IP) is **NP**-hard in almost all cases when $X = \{0, \dots, a-1\}^n$.

1.1. Previous work

The approximability of the program (IP) has been extensively studied in the case when A is restricted to non-negative entries. In this case, the problem is usually referred to as a (generalised, or capacitated) *covering problem*. Among the problems described by such programs one finds the Minimum Knapsack Problem, Minimum Vertex Cover (and its k-uniform hypergraph counterpart, described be-

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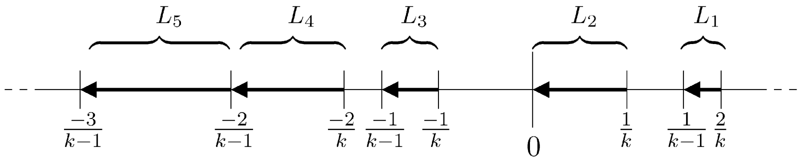

Fig. 1. The intervals $L_1, \ldots, L_5$ represented by arrows.

low) and various *network design problems* [2]. We will refer to (IP) with non-negative A as (CIP) (covering integer program). Here, X is often taken to be $\{\mathbf{x} \in \mathbb{Z}^n \mid \mathbf{0} \leqslant \mathbf{x} \leqslant \mathbf{d}\}$. Indeed, optimal solutions remain feasible after introduction of the bounds $x_i \leqslant \lceil \max_j b_j / a_{ji} \rceil$.

Hall and Hochbaum [7] restrict A in (CIP) to a 0/1-matrix and give an $\|A\|_\infty$-approximating algorithm for the case when $X = \{0,1\}^n$. Bertsimas and Vohra [1] study the general (CIP) with $X = \{0,1\}^n$ as well as $X = \mathbb{N}^n$. They use both a randomised rounding heuristic with a nonlinear rounding function and deterministic rounding using information about the dual program. For $X = \{0,1\}^n$, they show that (CIP) can be approximated within $\|A\|_\infty$ using both a deterministic rounding function and a dual heuristic. For $X = \mathbb{N}^n$, they obtain an $\|A\|_\infty + 1$ approximating algorithm. Carr, Fleischer, Leung and Phillips [2] lower the integrality gap of (CIP) with $X = \{0,1\}^n$ by introducing additional inequalities into the program to obtain an approximation ratio equal to the maximal number of non-zero entries in a row of A. Their claim that the proof immediately generalises to the case when the variables are bounded by any fixed $d > 1$ seems to be incorrect, but a complete proof for general d is given by Pritchard [11]. Koufogiannakis and Young [10] present an approximation algorithm for a general framework of *monotone covering* problems, with an approximation ratio equal to the maximal number of variables upon which a constraint depends. The constraints must be monotone (closed upwards), but can be non-convex. This framework in particular includes problems such as (CIP) and Minimum Set Cover.

2. Unbounded domain

We assume that $X = \mathbb{N}^n$ throughout this section. Lower bounds for $(\mathrm{IP})_k$ are discussed in Section 2.1. We aim to prove the following result:

Proposition 2.1. *$(\mathrm{IP})_k$ can be approximated within k.*

The problem $(\mathrm{IP})_1$ is solvable in polynomial time: initially, let $x_i = 0$ for all i, and for each inequality $x_i \geqslant b$, update x_i to $\max\{x_i, b\}$. Any inequality of the form $-x_i \geqslant b$ implies that there are no solutions. In order to prove Proposition 2.1 for $k \geqslant 2$, we give a deterministic 'rounding'-scheme, which produces an integer solution from a rational one, while increasing the value of the objective function by at most k. For an integer $k \geqslant 2$ and $x \in \mathbb{Q}^+ \cup \{0\}$, define the following operation:

$$\hat{x} = \begin{cases} 0 & \text{if } 0 \leqslant x < 1/k, \\ 1 & \text{if } 1/k \leqslant x < 2/k, \\ \lceil (k-1)x \rceil & \text{otherwise.} \end{cases}$$

For a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$, let $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)^T$. Note that $\mathbf{c}^T\hat{\mathbf{x}} \leqslant k \cdot \mathbf{c}^T\mathbf{x}$. We will show that in addition, $\hat{\mathbf{x}}$ satisfies $A\hat{\mathbf{x}} \geqslant \mathbf{b}$ by showing that for any integer $b \geqslant 1$, we have $\mathbf{a} \cdot \hat{\mathbf{x}} \geqslant b$ whenever $\mathbf{a} \cdot \mathbf{x} \geqslant b$ for any vector $\mathbf{a} = (a_1, \ldots, a_n)^T$ with $\|\mathbf{a}\|_1 \leqslant k$. In order to do this, we first introduce a scaling of $\hat{x}$ which will be easier to work with. Let $x' = \hat{x}/(k-1)$ and extend to vectors, $\mathbf{x}' = (x'_1, x'_2, \ldots, x'_k)^T$, as before.

Our first step is to bound the difference $\Delta = \mathbf{a} \cdot \mathbf{x} - \mathbf{a} \cdot \mathbf{x}'$ from above. Let $\delta_i = a_i(x_i - x'_i)$ so that $\Delta = \sum_{i=1}^n \delta_i$. Let $t_i = \mathrm{sgn}(a_i) \cdot x_i$ and $t'_i = \mathrm{sgn}(a_i) \cdot x'_i$. Then, $\delta_i = |a_i|(t_i - t'_i)$. Fig. 1 illustrates how the t'_i are determined from the t_i in the cases which give positive contributions to Δ. Each arrow represents an interval, and for a t_i in a particular interval, t'_i can be found at the arrow head. Note that there are only two such intervals on the positive axis. To the left of L_5 follows an infinite sequence of left arrows, each of size equal to that of L_5.

Formally, the intervals $L_i, i \geqslant 1$, are defined as follows:

$$L_1 = \{x \in \mathbb{Q} \mid 1/(k-1) \leqslant x < 2/k\},$$
$$L_2 = \{x \in \mathbb{Q} \mid 0 \leqslant x < 1/k\},$$
$$L_3 = \{x \in \mathbb{Q} \mid -1/(k-1) < x \leqslant -1/k\},$$
$$L_4 = \{x \in \mathbb{Q} \mid -2/(k-1) \leqslant x \leqslant -2/k\},$$
$$L_i = \{x \in \mathbb{Q} \mid -(i-2)/(k-1) \leqslant x < -(i-3)/(k-1)\} \quad (i \geqslant 5).$$

When $k = 2$, the interval L_1 vanishes while L_3 and L_4 become adjacent. Let $L = \bigcup_{i \geqslant 1} L_i$. Now, δ_i can be bounded as follows, given the location of t_i:

$$\begin{cases} 0 \leqslant \delta_i/|a_i| < (k-2)/k(k-1) & \text{if } t_i \in L_1, \\ 0 \leqslant \delta_i/|a_i| < 1/k & \text{if } t_i \in L_2, \\ 0 \leqslant \delta_i/|a_i| \leqslant 1/k(k-1) & \text{if } t_i \in L_3, \\ 0 \leqslant \delta_i/|a_i| \leqslant 2/k(k-1) & \text{if } t_i \in L_4, \\ 0 \leqslant \delta_i/|a_i| < 1/(k-1) & \text{if } t_i \in L_j,\ j \geqslant 5, \\ \delta_i \leqslant 0 & \text{if } t_i \notin L. \end{cases}$$

Note that when $k = 2$, the upper bound on $\delta_i/|a_i|$ for $t_i \in L_4$ is actually strict, since $-2/k$ is an integer. Thus, $\delta_i < |a_i|/(k-1)$, for all $i \geqslant 1$.

Lemma 2.2. *Let $b \geqslant 1$ and $k \geqslant 2$ be integers. If $\mathbf{a} \cdot \mathbf{x} \geqslant b$ and $\|\mathbf{a}\|_1 \leqslant k$, then $\Delta < 1$.*

Proof. Assume that there is an index l such that $t_l \notin L$. Then, $|a_l| > 0$ so $\sum_{i \neq l} |a_i| \leqslant k-1$. We then have

$$\Delta \leqslant \sum_{i \neq l} \delta_i < \sum_{i \neq l} \frac{|a_i|}{k-1} \leqslant \frac{k-1}{k-1} = 1. \tag{1}$$