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Vertex fault tolerance of optimal-κ graphs and super-κ graphs ☆

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ABSTRACT

A connected graph G is optimal-κ if $\kappa(G) = \delta(G)$. It is super-κ if every minimum vertex cut isolates a vertex. An optimal-κ graph G is m-optimal-κ if for any vertex set $S \subseteq V(G)$ with $|S| \leqslant m$, $G - S$ is still optimal-κ. We define the vertex fault tolerance with respect to optimal-κ, denoted by $O_\kappa(G)$, as the maximum integer m such that G is m-optimal-κ. The concept of vertex fault tolerance with respect to super-κ, denoted by $S_\kappa(G)$, is defined in a similar way. In this paper, we show that $\min\{\kappa_1(G) - \delta(G), \delta(G) - 1\} \leqslant O_\kappa(G) \leqslant \delta(G) - 1$ and $\min\{\kappa_1(G) - \delta(G) - 1, \delta(G) - 1\} \leqslant S_\kappa(G) \leqslant \delta(G) - 1$, where $\kappa_1(G)$ is the 1-extra connectivity of G. Furthermore, when the graph is triangle free, more refined lower bound can be derived for $O_\kappa(G)$.

1. Introduction

Throughout this paper, all graphs are simple. Let G be a connected graph. A vertex subset $S \subseteq V(G)$ is a *vertex cut* if $G - S$ is disconnected. The minimum cardinality of a vertex cut of G is called the *connectivity* of G, denoted by $\kappa(G)$. A vertex cut S with $|S| = \kappa(G)$ is called a *κ-cut*. In general, the larger $\kappa(G)$ is, the more reliable the graph is. Since $\kappa(G) \leqslant \delta(G)$, where $\delta(G)$ is the minimum degree of G, a connected graph G with $\kappa(G) = \delta(G)$ is said to be *maximally connected* (or *optimal-κ* for short). It is *super-connected* (*super-κ* for short) if every minimum vertex cut of G isolates a vertex. A super-κ graph is clearly optimal-κ.

In this paper, we are interested in the vertex fault tolerance with respect to optimal-κ and super-κ, the concept of which is defined in the following.

Definition 1.1. An optimal-κ (resp. super-κ) graph G is *m-optimal-κ* (resp. *m-super-κ*) if $G - S$ is still optimal-κ (resp. super-κ) for any vertex set $S \subseteq V(G)$ with $|S| \leqslant m$. We define the *vertex fault tolerance with respect to optimal-κ* (resp. *super-κ*), denoted by $O_\kappa(G)$ (resp. $S_\kappa(G)$), as the maximum integer m such that G is m-optimal-κ (resp. m-super-κ).

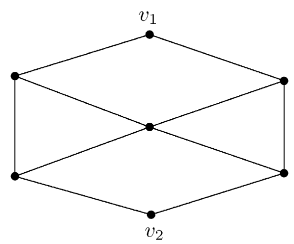

Fig. 1. A graph G with $O_\kappa(G) = 1$ and $S_\kappa(G) = 0$.

The graph G in Fig. 1 has $O_\kappa(G) = 1$ (notice that $G - \{v_1, v_2\}$ is no longer optimal-κ) and $S_\kappa(G) = 0$ (notice that $G - \{v_1\}$ is no longer super-κ).

The two concepts in Definition 1.1 generalize those of optimal-κ and super-κ (the special case when $m = 0$). In this paper, we study bounds for $O_\kappa(G)$ and $S_\kappa(G)$.

A related work is [7], where Hong and Meng first proposed the concept of edge fault tolerance for super edge connected graphs. A graph is *super edge connected* if ev-

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ery minimum edge cut isolates a vertex. A super edge connected graph G is *m-super edge connected* if $G-S$ is still super edge connected for any edge set $S \subseteq E(G)$ with $|S| \leqslant m$. The maximum integer of such m, denoted by $S_\lambda(G)$, is the *edge fault tolerance with respect to super edge connectedness*. In [7], the authors showed that $\{\lambda'(G)-\delta(G)-1, \delta(G)-1\} \leqslant S_\lambda(G) \leqslant \delta(G)-1$, where $\lambda'(G)$ is the restricted edge connectivity of G (which was first proposed by Esfahanian and Hakimi in [2]). More refined bounds are given for regular graphs and Cartesian product graphs. Furthermore, exact value of $S_\lambda(G)$ is determined for edge transitive graph.

Although our bounds for $O_\kappa(G)$ and $S_\kappa(G)$ are similar to those for $S_\lambda(G)$ in [7], the derivation is much more complicated. For this purpose, we need the concept of extra connectivity first proposed by Fàbrega and Fiol [4,5]. A vertex set S of a connected graph G is an *i-extra cut* if each component of $G-S$ has order at least $i+1$. The minimum cardinality of all i-extra cuts (if any) is the *i-extra connectivity* of G, denoted by $\kappa_i(G)$. An i-extra cut S with $|S| = \kappa_i(G)$ is called a *κ_i-cut*. In general, $\kappa_i(G)$ does not always exist and the graphs in which $\kappa_i(G)$ exist are said to be *κ_i-connected*. For a graph G which is not κ_i-connected, we define $\kappa_i(G) = \infty$. By the definition, it is easy to see that $\kappa_i(G)$ is monotone non-decreasing in i, that is, $\kappa_1(G) \leqslant \kappa_2(G) \leqslant \kappa_3(G) \leqslant \cdots$.

In this paper, we show that $\min\{\kappa_1(G)-\delta(G), \delta(G)-1\} \leqslant O_\kappa(G) \leqslant \delta(G)-1$ and $\min\{\kappa_1(G)-\delta(G)-1, \delta(G)-1\} \leqslant S_\kappa(G) \leqslant \delta(G)-1$. Furthermore, when the graph is triangle free, more refined lower bound can be derived for $O_\kappa(G)$ in terms of $\kappa_i(G)$.

Next, we introduce some notion which will be used in this paper. For two disjoint vertex sets $U_1, U_2 \subset V(G)$, denote by $[U_1, U_2]_G$ the set of edges of G with one end in U_1 and the other end in U_2. For a vertex set $U \subseteq V(G)$, $G[U]$ is the subgraph of G induced by U, $\overline{U} = V(G) \setminus U$ is the complement of U, $\omega_G(U) = |[U, \overline{U}]_G|$ is the number of edges between U and $\overline{U}$, $N_G(U) = \{v \in V(G) \setminus U \mid v$ is adjacent with some vertex in $U\}$ is the *neighborhood* of U, $N_G[U] = N_G(U) \cup U$ is the *the closed neighborhood of* U. If U has exactly one vertex v, we use $N_G(v)$ instead of $N_G(\{v\})$, etc. The *degree* of a vertex v in G is $d_G(v) = |N_G(v)|$. When the graph under consideration is obvious, we use $N(U)$, δ, etc. instead of $N_G(U)$, $\delta(G)$, etc.

For more studies on connectivity of graphs, we refer the reader to survey articles by Fàbrega and Fiol [3], Mader [8], Oellermann [9] and Hellwig and Volkmann [6]. For terminology not given here, we refer [1] for references.

2. Bounds for $O_\kappa(G)$

For a vertex set $S \subseteq V(G)$, to measure whether $G-S$ is optimal-κ, we need the following necessary and sufficient condition for a graph to be optimal-κ.

Lemma 2.1. *A connected graph G is optimal-κ if and only if $|N(X)| \geqslant \delta(G)$ for any non-empty vertex set X with $V(G) \setminus (X \cup N(X)) \neq \emptyset$.*

Proof. In fact, $\kappa(G) = \min\{|N(X)|: X \subset V(G), V(G) \setminus (X \cup N(X)) \neq \emptyset\}$. The lemma follows from the definition of optimal-κ. □

The following lemma is an easy observation.

Lemma 2.2. *Let S, X be two subsets of $V(G)$ with $X \nsubseteq S$. Then $N_G(X) - S \supseteq N_{G-S}(X-S)$. Furthermore, equality holds if $S \cap X = \emptyset$.*

If $G-S$ is not optimal-κ, then by Lemma 2.1, there exists a non-empty vertex set $X \subseteq V(G)-S$ such that $|N_{G-S}(X)| < \delta(G-S)$ and $N_{G-S}(X) \cup X \neq V(G-S)$. The next lemma characterizes a special set of such an X.

Lemma 2.3. *For a vertex set $S \subseteq V(G)$, suppose X is a non-empty vertex set such that*

(a) *$X \subseteq V(G)-S$, $|N_{G-S}(X)| < \delta(G-S)$, $N_{G-S}(X) \cup X \neq V(G-S)$, and*
(b) *under the condition of (a), $|X|$ is as small as possible.*

Then

(i) *$G[X]$ is connected;*
(ii) *for any component C of $G-N(X)$ with $|V(C)| < |X|$, $V(C) \subseteq S$;*
(iii) *$|X| \geqslant 2$;*
(iv) *for any $y \in N_{G-S}(X)$, $|N_{G-S}(y) \cap X| \geqslant 2$.*

Proof. (i) Suppose that $G[X]$ is not connected. Let C be a component of $G[X]$. Then $V(C)$ is a non-empty vertex set of $G-S$ with $N_G(V(C)) \subseteq N_G(X)$. By Lemma 2.2, we have

$$N_{G-S}\big(V(C)\big) = N_G\big(V(C)\big) - S \subseteq N_G(X) - S = N_{G-S}(X). \tag{1}$$

It follows that $|N_{G-S}(V(C))| \leqslant |N_{G-S}(X)| < \delta(G-S)$ and $N_{G-S}(V(C)) \cup V(C) \subseteq N_{G-S}(X) \cup X \neq V(G-S)$. Hence $V(C)$ is a smaller non-empty vertex set satisfying condition (a), contradicting condition (b).

(ii) The proof is similar to that of (i) by showing that if $V(C) \nsubseteq S$, then $X_1 = V(C) - S$ is a smaller non-empty set satisfying condition (a). There are two differences here. The first is that $N_{G-S}(X_1) \subseteq N_G(V(C)) - S$ (by Lemma 2.2) is used to replace the first equality of Eq. (1); the second is that the third condition of (a) is satisfied by noting that $X \subseteq V(G-S) - (N_{G-S}(X_1) \cup X_1)$ is non-empty.

(iii) Suppose X has only one vertex x. Then $|N_{G-S}(X)| = d_{G-S}(x) \geqslant \delta(G-S)$, contradicting condition (a).

(iv) Suppose there exists a vertex $y \in N_{G-S}(X)$ such that $|N_{G-S}(y) \cap X| = 1$. Suppose x is the only vertex in $N_{G-S}(y) \cap X$. Set $X_1 = X - \{x\}$. By (i), we see that $x \in N_G(X_1)$. By Lemma 2.2 and the observation $N_G(X_1) \subseteq N_G(X) \cup \{x\} - \{y\}$, we have $|N_{G-S}(X_1)| = |N_G(X_1) - S| \leqslant |N_G(X) \cup \{x\} - \{y\} - S| = |N_G(X) - S| = |N_{G-S}(X)| < \delta(G-S)$. Furthermore, $N_{G-S}(X_1) \cup X_1 = (N_G(X_1) - S) \cup X_1 \subseteq (N_G(X) \cup \{x\} - \{y\} - S) \cup (X - \{x\}) \subseteq N_{G-S}(X) \cup X \neq V(G-S)$. A contradiction occurs as before. □