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Hardness of conjugacy, embedding and factorization of multidimensional subshifts ☆

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ABSTRACT

Subshifts of finite type are sets of colorings of the plane defined by local constraints. They can be seen as a discretization of continuous dynamical systems. We investigate here the hardness of deciding factorization, conjugacy and embedding of subshifts in dimensions $d > 1$ for subshifts of finite type and sofic shifts and in dimensions $d \geq 1$ for effective shifts. In particular, we prove that the conjugacy, factorization and embedding problems are Σ_3^0-complete for sofic and effective subshifts and that they are Σ_1^0-complete for SFTs, except for factorization which is also Σ_3^0-complete.

A d-dimensional subshift is the set of colorings of $\mathbb{Z}^d$ by a finite set of colors in which a family of forbidden patterns never appear. These are shift-invariant spaces, hence the name. If the family of forbidden patterns is finite, then it is a subshift of finite type (SFT). If the family of forbidden patterns is recursively enumerable, then the subshift is called effective. Another class of subshifts can be defined by the help of local maps, namely the class of sofic shifts: they are the letter by letter projections of SFTs.

One can also see SFTs as tilings of $\mathbb{Z}^d$, and in dimension 2 they are equivalent to the usual notion of tilings introduced by Wang [17]. Subshifts are a way to discretize continuous dynamical systems: if X is a compact space and $\phi : X \to X$ a continuous map, we can partition X in a finite number of parts $\mathcal{A} = \{1, \dots, n\}$ and transform the orbit of a point $x \in X$ into a sequence $(x_n)_{n\in\mathbb{N}^*}$, where x_i denotes the part of X in which $\phi^i(x)$ lies.

Conjugacy is the right notion of isomorphism between subshifts, and plays a major role in their study: when two subshifts are conjugate they code each other and hence have the same dynamical properties. Conjugacy is an equivalence relation and allows to separate SFTs into equivalence classes. Deciding whether two SFTs are conjugate is called the classification problem. It is a long standing open problem in dimension one [5], although has been proved decidable in the particular case of one-sided SFTs on $\mathbb{N}$, see [18]. It has been known for a long time that in higher dimensions the problem is undecidable when given two SFTs, since it can be reduced to the emptiness problem which is Σ_1^0-complete [2]. However,

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we prove here a slightly stronger result: even by fixing the class in advance, it is still undecidable to decide whether some given SFT belongs to it:

Theorem 0.1. *For any fixed SFT X, given some SFT Y as an input, it is Σ_1^0-complete to decide whether X and Y are conjugate (resp. equal).*

As for the classes of sofic and effective shifts, the complexity is higher:

Theorem 0.2. *Given two sofic/effective shifts X, Y, it is Π_2^0-complete to decide whether X and Y are equal.*

Theorem 0.3. *Given two sofic/effective subshifts X, Y, it is Σ_3^0-complete to decide whether X and Y are conjugate.*

An interesting open question for higher dimension that would probably help solve the one dimensional problem would be *is conjugacy of subshifts decidable when provided an oracle answering whether or not a pattern is extensible?* A positive answer to this question would solve the one dimensional case, even if the SFTs are considered on $\mathbb{N}^2$ instead of $\mathbb{Z}^2$.

Factorization is the notion of surjective morphism adapted to SFTs: when X factors on Y, then Y is a recoding of X, possibly with information loss: the dynamic of Y is "simpler" than X's, i.e. it can be deduced from X's. The problem of knowing if some SFT is a factor of another one has also been much studied. In dimension one, it is only partly solved for the case when the entropies of the two SFTs X, Y verify $h(X) > h(Y)$, see [4]. Factor maps have also been studied with the hope of finding universal SFTs: SFTs that can factor on any other and thus contain the dynamics of all of them. However it has been shown that such SFTs do not exist, see [7,3]. We prove here that it is harder to know if an SFT is a factor of another than to know if it is conjugate to it.

Theorem 0.4. *Given two SFTs/sofic/effective subshifts X, Y as input, it is Σ_3^0-complete to decide whether X factors onto Y.*

The last problem we will tackle is the embedding problem, that is to say: when can an SFT be injected into some other SFT? If an SFT X can be injected into another SFT Y, that means that there is an SFT $Z \subseteq Y$ such that X and Z are conjugate. In dimension 1, this problem is also partly solved when the two SFTs X, Y are irreducible and their entropies verify $h(X) > h(Y)$ [12]. We prove here that the problem is Σ_1^0-complete for SFTs and Σ_3^0-complete for effective and sofic subshifts:

Theorem 0.5. *Given two SFTs X, Y as inputs, it is Σ_1^0-complete to decide whether X embeds into Y.*

Theorem 0.6. *Given two sofic/effective subshifts X, Y as inputs, it is Σ_3^0-complete to decide whether X embeds into Y.*

The paper is organized as follows: first we give the necessary definitions and fix the notation in Section 1, after what we give the proofs of the theorems about conjugacy and equality in Section 2, about factorization in Section 3 and about embedding in Section 4.

This article covers the results announced in [10] with the additions of the results on sofic and effective subshifts.

1. Preliminary definitions

1.1. SFTs and effective subshifts

We give here some standard definitions and facts about multidimensional subshifts, one may consult Lind [14] or Lind/Marcus [13] for more details.

Let $\mathcal{A}$ be a finite alphabet, its elements are called *symbols*, the d-dimensional full shift on $\mathcal{A}$ is the set $\mathcal{A}^{\mathbb{Z}^d}$ of all maps (colorings) from $\mathbb{Z}^d$ to the $\mathcal{A}$ (the colors). For $v \in \mathbb{Z}^d$, the shift functions $\sigma_v : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$, are defined locally by $\sigma_v(c_x) = c_{x+v}$. The full shift equipped with the distance $d(x, y) = 2^{-\min\left\{\|v\| \middle| v \in \mathbb{Z}^d, x_v \neq y_v\right\}}$ is a compact metric space on which the shift functions act as homeomorphisms. An element of $\mathcal{A}^{\mathbb{Z}^d}$ is called a *configuration*.

Every closed shift-invariant (invariant by application of any σ_v) subset X of $\mathcal{A}^{\mathbb{Z}^d}$ is called a *subshift*, or *shift*. An element of a subshift is called a *point* of this subshift.

Alternatively, subshifts can be defined with the help of forbidden patterns. A *pattern* is a function $p : P \to \mathcal{A}$, where P, the *support*, is a finite subset of $\mathbb{Z}^d$. Let $\mathcal{F}$ be a collection of *forbidden* patterns, the subset $X_\mathcal{F}$ of $\mathcal{A}^{\mathbb{Z}^d}$ containing the configurations having nowhere a pattern of $\mathcal{F}$. More formally, $X_\mathcal{F}$ is defined by

$$X_\mathcal{F} = \left\{x \in \mathcal{A}^{\mathbb{Z}^d} \middle| \forall z \in \mathbb{Z}^d, \forall p \in \mathcal{F}, x_{|z+P} \neq p\right\}.$$