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Approximating the partition function of planar two-state spin systems [☆]

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A B S T R A C T

We consider the problem of approximating the partition function of the hard-core model on planar graphs of degree at most 4. We show that when the activity λ is sufficiently large, there is no fully polynomial randomised approximation scheme for evaluating the partition function unless NP = RP. The result extends to a nearby region of the parameter space in a more general two-state spin system with three parameters. We also give a polynomial-time randomised approximation scheme for the logarithm of the partition function.

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1. Introduction

A spin system is a model of particle interaction on a graph. Every vertex of the graph is assigned a state, called a spin. A *configuration* assigns a spin to every vertex, and the weight of the configuration is determined by interactions of neighbouring spins.

In this paper, we consider the following two-spin model, which applies to spin systems on a graph $G = (V, E)$. The model has three parameters, β, γ and λ. It is easiest to view these as non-negative rationals for now — we will be slightly more general later. A configuration $\sigma : V(G) \to \{0, 1\}$ is an assignment of the two spins “0” and “1” to the vertices in V. The configuration σ has a *weight* $w_G(\sigma)$, which depends upon β, γ and λ. Let $b(\sigma)$ denote the number of edges (u, v) of G with $\sigma(u) = \sigma(v) = 0$, let $c(\sigma)$ be the number of edges (u, v) of G with $\sigma(u) = \sigma(v) = 1$ and let $\ell(\sigma)$ be the number of vertices u of G with $\sigma(u) = 1$. Then $w_G(\sigma) = \beta^{b(\sigma)}\gamma^{c(\sigma)}\lambda^{\ell(\sigma)}$. The *partition function* of the model is given by

$$Z_{\beta,\gamma,\lambda}(G) = \sum_{\sigma : V(G)\to\{0,1\}} w_G(\sigma).$$

Two important special cases are

- the case $\beta = 1$, $\gamma = 0$, which is the *hard-core model*, and
- the case $\beta = \gamma$, which is the *Ising model*.

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The *hard-core model* [3] is a model of a gas in which vertices are either occupied by a particle (in which case they have spin 1) or unoccupied (in which case they have spin 0). The particles cannot overlap and adjacent vertices are close together, hence $\gamma = 0$. The Ising model is a model of ferromagnetism. In this paper we study the hard-core model and a region of nearby two-state spin systems.

1.1. Previous work

Evaluating $Z_{\beta,\gamma,\lambda}(G)$ is a trivial computational problem if $\beta\gamma = 1$, because the partition function factors. In other cases, the complexity of evaluation has been studied in detail. When $\lambda = 1$, the problem of computing the partition function on planar Δ-regular graphs is called $\text{Pl-Hol}_\Delta(a, b)$ in [7], where a corresponds to β and b corresponds to γ. Assume $\Delta \geq 3$. There is a dichotomy [7, Theorem 1]: for non-negative a, b, the problem $\text{Pl-Hol}_\Delta(a, b)$ can be computed exactly in polynomial time in the trivial cases $ab = 1$ and $a = b = 0$, and in the case of the Ising model with no external field, $a = b$. In all other cases, the problem of exactly computing the partition function is #P-hard.

A standard transformation extends this dichotomy to arbitrary $\lambda > 0$. Consider a configuration $\sigma : V(G) \to \{0, 1\}$ of a planar Δ-regular graph G. Counting the number of edges adjacent to a "1" spin in two ways, we have $\Delta\ell(\sigma) = 2c(\sigma) + (|E(G)| - b(\sigma) - c(\sigma))$. Therefore,

$$Z_{\beta,\gamma,\lambda}(G) = \lambda^{|E(G)|/\Delta} Z_{\beta\lambda^{-1/\Delta},\gamma\lambda^{1/\Delta},1}(G),$$

which is as hard to compute as $\text{Pl-Hol}_\Delta(\beta\lambda^{-1/\Delta}, \gamma\lambda^{1/\Delta})$. Suppose β and γ are not both 0. Unless $\lambda = 1$, we have either $\beta\lambda^{-1/3} \neq \gamma\lambda^{1/3}$ or $\beta\lambda^{-1/4} \neq \gamma\lambda^{1/4}$. If $\beta\gamma \neq 1$ then in either case, we can conclude from above that evaluating $Z_{\beta,\gamma,\lambda}(G)$ is #P-hard when the input G is restricted to be a planar graph of degree at most 4.

Since the complexity of exactly evaluating the partition function is intractable, much effort has focused on the difficulty of *approximately* evaluating the partition function for a given set of parameters β, γ and λ.

The complexity of approximating the partition function of the hard-core model and the Ising model in general (not necessarily planar) graphs is well-understood. The *Gibbs measure* is the distribution on configurations $\sigma : V(G) \to \{0, 1\}$ in which the probability of configuration σ is proportional to $w_G(\sigma)$. This notion of Gibbs measure extends to certain infinite graphs, for example infinite regular trees, where it may or may not be unique. For the hard-core model, there is a critical point $\lambda_c(\Delta) = (\Delta - 1)^{\Delta-1}/(\Delta - 2)^\Delta$ such that the infinite Δ-regular tree has a unique Gibbs measure if and only if $\lambda \leq \lambda_c$. An important result of Weitz [26] showed that, in *every* graph with maximum degree at most Δ, the correlations between spins in the hard-core model decay rapidly with distance as long as $\lambda \leq \lambda_c$. As a result, he gives [26, Corollary 2.8] a fully-polynomial (deterministic) approximation scheme (FPTAS) for evaluating the hard-core partition function on graphs of degree at most Δ for any $\lambda < \lambda_c$. By contrast, Sly and Sun [24, Theorem 1] (see also the earlier hardness results of Sly [25] and Galanis et al. [10]) show that, unless NP = RP, there is no fully-polynomial randomised approximation scheme (FPRAS) on Δ-regular graphs (for $\Delta \geq 3$) for any $\lambda > \lambda_c(\Delta)$. Thus, the difficulty of approximation is resolved, apart from at the boundary $\lambda = \lambda_c(\Delta)$.

We say that the two-spin model is *ferromagnetic* if $\beta\gamma > 1$ and *antiferromagnetic* if $\beta\gamma < 1$. For the antiferromagnetic Ising model, Sinclair et al. [23, Corollary 1] show that there is an FPTAS for evaluating the Ising partition function on graphs of degree at most Δ for any choice of parameters β and λ which is in the interior of the uniqueness region of the Δ-ary tree. By contrast, Sly and Sun [24, Theorem 2] show that, unless NP = RP, there is no FPRAS on Δ-regular graphs (for $\Delta \geq 3$) if β and λ are outside the uniqueness region. (So, once again, the situation is fully resolved, apart from the boundary.) The result of Sinclair et al. extends to general anti-ferromagnetic two-state spin systems in regular graphs, and also in a somewhat wider class of graphs [23, Corollary 2].

For general anti-ferromagnetic two-state spin systems, the best positive result that is known is due to Li, Lu, and Yin [19]. They use a stronger notion of correlation decay than Weitz, which enables them to obtain a PTAS, even for graphs with unbounded degree. They show [19, Theorem 2] that for any finite $\Delta \geq 3$, or for $\Delta = \infty$, there is an FPTAS for the partition function of the two-state spin system on graphs of maximum degree at most Δ if the parameters of the system are antiferromagnetic, and for every $d \leq \Delta$, they lie in the interior of the uniqueness region of the infinite d-regular tree. By contrast [19, Theorem 3], the results of Sly and Sun imply that, for any finite $\Delta \geq 3$, or for $\Delta = \infty$, unless NP = RP, there is no FPRAS for the partition function of the two-state spin system on graphs of maximum degree at most Δ if the parameters of the system are antiferromagnetic, and for some $d \leq \Delta$, they lie outside the interior of the uniqueness region of the infinite d-regular tree. Thus, the approximation complexity is resolved in the antiferromagnetic case, apart from at the boundaries of the uniqueness regions. Note that the result of Sun and Sly was independently discovered by Galanis, Štefankovič and Vigoda [11] for the case $\lambda = 1$.

The situation is not completely resolved in the ferromagnetic case. Building on Jerrum and Sinclair's FPRAS for the ferromagnetic Ising model [16], Goldberg, Jerrum and Paterson [15] gave an FPRAS for the ferromagnetic two-spin model which applies if $\beta \geq \gamma$ and $\lambda \leq \sqrt{\beta/\gamma}$ (or, equivalently, if $\beta \leq \gamma$ and $\lambda \geq \sqrt{\beta/\gamma}$). The approximation applies without these constraints on the parameters if the input is a regular graph.

For the hard-core model, an important issue which arises in statistical physics is approximating the partition function for planar graphs, including regular lattices. While (as far as we know) there were no hardness results for this problem (until this paper) the complexity of particular algorithms has been studied. For example, Randall [21] showed that a particular

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