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Quadpack computation of Feynman loop integrals

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ABSTRACT

The paper addresses a numerical computation of Feynman loop integrals, which are computed by an extrapolation to the limit as a parameter in the integrand tends to zero. An important objective is to achieve an automatic computation which is effective for a wide range of instances. Singular or near singular integrand behavior is handled via an adaptive partitioning of the domain, implemented in an iterated/repeated multivariate integration method. Integrand singularities possibly introduced via infrared (IR) divergence at the boundaries of the integration domain are addressed using a version of the DQAGS algorithm from the integration package Quadpack, which uses an adaptive strategy combined with extrapolation. The latter is justified for a large class of problems by the underlying asymptotic expansions of the integration error. For IR divergent problems, an extrapolation scheme is presented based on dimensional regularization.

1. Introduction

This article describes applications of numerical integration and extrapolation for the computation of some classes of Feynman integrals. Sample results are included for one-loop pentagon and two-loop ladder box diagrams and a new approach is presented for handling infrared divergence using dimensional regularization and extrapolation.

The motivations of high energy physics collider experiments include the precise measurement of parameters in the standard model [1,2] and detection of any deviations of experimental data from the theoretical prediction, leading to the study of new phenomena.

The computation of loop integrals is required in high energy physics to obtain higher order terms in perturbation calculations of the scattering amplitude. Generally, with a given set of external particles and the interaction, a large number of Feynman diagrams is associated. Each diagram represents one of the possible configurations of virtual processes and it describes a part of the amplitude. The square sum of the amplitudes gives the probability or cross section of the process.

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Feynman integrals suffer from singularities through vanishing integrand denominators. The momentum integration of loop integrals is performed with Feynman's parametrization technique. Ultraviolet poles, due to the integrand behavior for large momenta, are removed by a renormalization procedure. Infrared (IR) poles may be present when some masses are negligible and give rise to non-integrable boundary singularities in the Feynman parametric form; their contribution can be eliminated via dimensional regularization. Sector decomposition [3–6] is applied to disentangle overlapping singularities. We give results in [7–9] using sector decomposition and iterated integration followed by extrapolation. An alternative approach introduces a fictitious small mass for the photon, as in [7,10]. Results using the extended precision package HMLIB [11] are reported in [10].

For physical kinematics, the parametrized form may have integrand singularities in the interior of the integration domain. Traditionally, numerical contour integration has been performed for singularities at specific locations in the integration region. The contour is moved away from the singularity in the complex plane, followed by a Monte-Carlo integration [12].

For an automatic computation without explicit information about the location or nature of the singularity, we have studied the application of extrapolation techniques to the computation of one-loop and two-loop diagrams. The latter include 3-point (*vertex*), 4-point (*box*) [13,14,8,10], 5-point (*pentagon*) and, after reductions, 6-point (*hexagon*) diagrams [15]; as well as two-loop *self-energy* [16], *double box* (*ladder*) and *crossed vertex* [17,18] diagrams. Results for a box diagram contribution with complex masses are given in [16].

We compute a numerical integral approximation as a limit where the value of a parameter in the integrand tends to zero. This involves the evaluation of a sequence of integrals for decreasing values of this parameter and a procedure for convergence acceleration or extrapolation to the limit.

As the integral computations often involve singularities within or at the boundaries of the integration region, a suitable numerical integration technique is essential. For an integral of a fairly low dimension, we use numerical iterated (repeated) integration where lower-dimensional adaptive methods are invoked recursively. Although we have, in several cases, combined one- and multi-dimensional integration methods in different sets of coordinate directions, repeated integration with 1D methods from QUADPACK [19] has been effective for many challenging numerical integration problems where standard multivariate integration software fails.

An outline of the QUADPACK general adaptive integration algorithms is provided in Section 3, as well as some numerical background on extrapolation methods. Definitions and notations pertaining to Feynman diagrams, Feynman parametrization and the corresponding integrals are given in Section 2. Section 4 gives extrapolation results for the one-loop pentagon diagram and Section 5 describes a computation for the two-loop (ladder) box.

IR-divergent problems are introduced in Section 6 and extrapolation results are included for a sample problem, as well as a massless vertex which is difficult numerically in view of its challenging singular behavior. Results for IR-singularities are obtained with a linear extrapolation, which is justified on the basis of the asymptotic integral expansions. The latter are generally expressed in terms of hypergeometric functions. An extrapolation approach for hypergeometric functions is further given in Section 7.

2. Feynman integral and parameter form

A Feynman diagram is a graph where each edge (or *line*) represents an intermediate state of a particle, and particles meet at the *vertices*. An edge may be incident to only one vertex, in which case it is called an *external line*; an *external vertex* is incident with at least one external line. Vertices and lines other than external ones are called *internal* (vertices and lines). The number of external lines equals the total number of particles present in the initial and final states [20]. A diagram may have one or multiple *loops* or cycles.

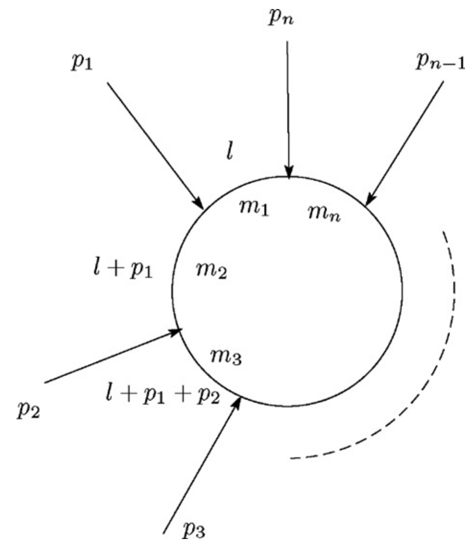

Fig. 1. One-loop diagram with n external legs, where p_ν is the incoming momentum of the ν-th external particle, m_j the mass carried by the j-th internal line and the momentum is $k_j = l + \sum_{\nu=1}^{j-1} p_\nu$.

Fig. 1 shows a one-loop diagram with n external legs [21], where p_ν is the momentum (directed inward) of the ν-th external particle. The j-th internal line associated with a spinless particle of mass m_j and 4-momentum $k_j = l + \sum_{\nu=1}^{j-1} p_\nu$ introduces a *Feynman propagator* of the form $i/(k_j^2 - m_j^2 + i\delta_j)$ into the Feynman integral, which will be calculated in the limit as $\delta_j \to 0$. In general, k_j-dependencies also enter in the numerator of the integrand. The one-loop (n-point) Feynman integral is expressed in [21] as

$$\mathcal{I}_n[\wp] = \int \frac{d^4 l}{\pi^2 i} \frac{\wp(k_1, \ldots, k_n)}{\prod_{j=1}^{n}(k_j^2 - m_j^2 + i\delta_\ell)} \quad (1)$$

where $\wp$ is a polynomial in the momenta k_j. The integral is called *scalar* if $\wp \equiv 1$.

The propagators introduce several types of singularities and the representation is not well defined mathematically due to the following reasons [20]: (a) Since the propagators are regarded as distributions [22], their product is not well defined; this problem is circumvented by regarding δ_j finite during the integration. (b) The integrations need to be performed in a specified manner. The Feynman parametrization is one of the techniques to perform the integration. (c) The contribution from very large momenta may cause *ultraviolet divergence*, which is dealt with by renormalization. To bypass the problem, the ultraviolet divergence can be separated by dimensional regularization. (d) *Infrared (IR) divergence* can occur with vanishing k_j when masses are zero; IR divergence cannot be removed from the transition amplitude but cancels out in the overall transition probability [23].

In order to allow numerical integration in cases with IR singularities, *dimensional regularization* is a technique to determine the parts of the integrand responsible for the divergence. The infinite IR contributions correspond to poles in a Laurent expansion of the integral with respect to ε, where $N - 2\varepsilon = 4$ and N replaces the integral dimension as $\varepsilon \to 0$.