Journal of Computer and System Sciences 82 (2016) 959–1006

Contents lists available at ScienceDirect

Journal of Computer and System Sciences

www.elsevier.com/locate/jcss

A logical approach to locality in pictures languages

Etienne Grandjean [a], Frédéric Olive [b,*]

[a] Université de Caen Normandie/ENSICAEN/CNRS, GREYC, Caen, France
[b] Aix-Marseille Université/CNRS, LIF, Marseille, France

ARTICLE INFO

Article history:
Received 26 July 2012
Received in revised form 6 May 2015
Accepted 22 January 2016
Available online 9 March 2016

Keywords:
Picture languages
Locality and tiling
Recognizability
Linear time
Cellular automata
Logical characterizations
Monadic second-order logic
Existential second-order logic

ABSTRACT

This paper deals with descriptive complexity of picture languages of any dimension by fragments of existential second-order logic:
1) We generalize to any dimension the characterization by Giammarresi et al. (1996) of the class of recognizable picture languages in existential monadic second-order logic.
2) We state natural logical characterizations of the class of picture languages of any dimension $d \geq 1$ recognized in linear time on nondeterministic cellular automata, a robust complexity class that contains, for $d = 1$, all the natural NP-complete problems.
Our characterizations are essentially deduced from normalization results for first-order and existential second-order logics over pictures.

Contents

* Corresponding author.
E-mail addresses: etienne.grandjean@unicaen.fr (E. Grandjean), frederic.olive@lif.univ-mrs.fr (F. Olive).

http://dx.doi.org/10.1016/j.jcss.2016.01.005

1. Introduction: context and discussion

Locality is a useful and widespread concept common to many areas of science: physics, chemistry, mathematics, etc. In computer science, it is a unifying notion, connecting combinatorics, logic, formal language theory, computational models, and complexity theory. For example, the local and combinatorial notion of *tiling* allowed Hao Wang et al. to prove in 1962 the undecidability of the decision problem of some logics [36,76,77,3]. Locality is also a reference notion in computational complexity (e.g., see [1,74,75]) and in formal language theory with the notion of *regular* or *recognizable* language that has been extended to tree or graph languages (see [70,6]). Typically, as recalled by Borchert [2], Mac Naughton and Papert established in their classical monograph [53] that a word language is regular "iff it consists of the words whose positions can be colored so that the coloring respects the letters and obeys a given finite set of neighborhood constraints".

There is a wealth of notions of locality in *logic* and *finite model theory*. For first-order logic, Libkin's book [46] (see Chapters 4 and 5) identifies *Hanf locality* [31] and *Gaifman locality* [17] and describes a series of locality results for this logic [12,15,63,32,64,45] and its order-invariant extension [30] or counting extension [44].

As a striking result, Gaifman's Theorem states in 1982 [17] that any first-order sentence is equivalent to a boolean combination of *local* sentences: roughly, a local sentence states the existence of k elements $x_1, \dots, x_k$, at distance $2d$ from each other (for some fixed d) such that for each x_i, the restriction of the structure to the set of elements at distance d of x_i has some fixed property ψ.

When applied to a class of structures of *bounded degree*, e.g. the class of cubic graphs, the local feature of first-order sentences can be even strengthened. As shown in [8,47], such a sentence is essentially equivalent to a boolean combination of cardinality formulas with only *one variable*, i.e. of the form $\exists^k x\, \psi(x)$, meaning "there exists k elements x that satisfy $\psi(x)$".

An even stronger notion of locality in logic is presented by Borchert in [2]. There, a picture language is *local* if it is defined as the set of pictures that do not contain any pattern belonging to some fixed finite set. Borchert proves that a picture language is local iff it is definable by a first-order sentence with only *one variable*, which is universally quantified, provided each picture is represented on its pixel domain with successor functions that encode the pixel adjacencies.

Computational models and *computational complexity* also involve several locality notions. Whereas it is questionable whether the Random Access Machine (RAM) or the pointer machine (e.g., see [61]) are local models, Turing machines and cellular automata are regarded as the prototypical models of *local* sequential and local parallel *computation*, respectively. Notice the *role of the underlying structure* for deciding what is local and what is nonlocal: while a configuration of a Turing machine or of a cellular automaton is essentially a word or a picture, that are *local structures*, a configuration of a RAM (resp. pointer machine) is a *function* from addresses to register contents (resp. from locations to locations). Clearly, such a function f allows to access in one step any location b from any other one a, even it they are *arbitrarily far* from each other, provided that $f(a) = b$: this contradicts the locality principle.

This paper[1] deals with *locality* in the context of *words* and *pictures* as underlying structures (e.g., see [2,18,19,21,22, 40–42,50,51,72]). For any dimension $d \geq 1$, a d-picture language is a set of d-dimensional words (colored d-dimensional grids). We study *descriptive complexity* of *nondeterministic* classes of word/picture languages by syntactical fragments of *existential second-order logic*. First, notice the following results:

1. In a series of papers culminating in [22], Giammarresi et al. proved that a 2-picture language is *recognizable* (i.e. is the projection of a local picture language) iff it is definable in *existential monadic logic* (EMSO). In short: $\text{REC}^2 = \text{EMSO}$. This is a picture language variant of the classical characterization of the regular/recognizable word language by (existential) monadic second-order logic, in short $\text{REG} = \text{REC}^1 = \text{EMSO} = \text{MSO}$ [4,10,73,53].
2. In fact, the class REC^2 contains some NP-complete problems. More generally, one observes that for each dimension $d \geq 1$, REC^d can be defined as the class of d-picture languages recognized in *constant time* by nondeterministic d-dimensional cellular automata. That means, for each $L \in \text{REC}^d$ there is some constant integer c such that each computation stops at instant c and a picture belongs to L iff it has at least one computation that stops *with each cell in an accepting state* (see e.g. [68]).

The present paper originates from two questions about word/picture languages:

- How can we generalize the proof of the above-mentioned theorem of Giammarresi et al. to any dimension? That is, can we establish the equality $\text{REC}^d = \text{EMSO}$ for d-picture languages of any dimension $d \geq 1$?
- Can we obtain logical characterizations of time complexity classes of cellular automata? This originates from a question Jacques Mazoyer asked the first author in 2001 (personal communication): exhibit a logical characterization of the linear time complexity class of nondeterministic cellular automata.

As Cris Moore has pointed to us (personal communication), it is significant that those picture language classes – recognizable languages and picture languages recognized by time bounded cellular automata – were invented *independently* in

[1] A preliminary and much shorter version of this paper appeared as a conference paper [28].